Weak Differential Marginality and the Shapley Value

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Abstract:
The principle of differential marginality for cooperative games states that the differential of two players’ payoffs does not change when the differential of these players’ productivities does not change. Together with two standard properties, efficiency and the null player property, differential marginality characterizes the Shapley value. For games that contain more than two players, we show that this characterization can be improved by using a substantially weaker property than differential marginality. Weak differential marginality requires two players’ payoffs to change in the same direction when these players’ productivities change by the same amount.
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Abstract

The principle of differential marginality for cooperative games states that the differential of two players’ payoffs does not change when the differential of these players’ productivities does not change. Together with two standard properties, efficiency and the null player property, differential marginality characterizes the Shapley value. For games that contain more than two players, we show that this characterization can be improved by using a substantially weaker property than differential marginality. Weak differential marginality requires two players’ payoffs to change in the same direction when these players’ productivities change by the same amount.

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1. Introduction

The Shapley value (Shapley, 1953) probably is the most eminent one-point solution concept for cooperative games with transferable utility (TU games). Besides its original axiomatic foundation by Shapley himself, alternative foundations of different types have been suggested later on. Important direct axiomatic characterizations are due to Myerson (1980) and Young (1985). Hart and Mas-Colell (1989) suggest an indirect characterization as the marginal contributions of a potential (function).\textsuperscript{1} Roth (1977) shows that the Shapley value can be understood as a von Neumann-Morgenstern utility. As a contribution to the Nash program, which aims at building bridges between cooperative and non-cooperative game theory, Pérez-Castrillo and Wettstein (2001) implement the Shapley value as the outcomes

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\textsuperscript{1}Calvo and Santos (1997) and Ortmann (1998) generalize the notion of a potential.

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of the sub-game perfect equilibria of a combined bidding and proposing mechanism, which is modeled by a non-cooperative extensive form game.\(^2\)

Among the one-point solution concepts for TU games, the Shapley value can be viewed as the measure of the players’ own productivity in a game. This view is strongly supported by Young (1985) characterization via three properties: efficiency, strong monotonicity, and symmetry.\(^3\) Efficiency says that the worth generated by the grand coalition is distributed among the players. Strong monotonicity requires a player’s payoff not to decrease whenever her productivity, measured by her marginal contributions to coalitions of the other players, weakly increases. Symmetry ensures that equally productive players obtain the same payoff.

In order to allow for solidarity among players, Casajus and Huettner (2013) suggest a differential version of strong monotonicity called differential monotonicity. This property requires two players’ payoff differential not to decrease whenever their productivity differential doesn’t decrease. Differential monotonicity is a strengthened version of differential marginality (Casajus, 2011), which demands equal productivity differentials to translate into equal payoff differentials.\(^4\) Any of the afore-mentioned properties together with efficiency and the null player property characterizes the Shapley value (van den Brink, 2001, Theorem 2.5; Casajus, 2011, Corollary 4).

In this paper, we consider a substantial relaxation of differential marginality, which we call weak differential marginality. Differential marginality can be rephrased as that equal changes in two players’ productivities should lead to equal changes in their payoff, which obviously implies that both payoffs change in the same direction. Weak differential marginality relaxes differential marginality in this vein. Equal changes in two players’ productivities should entail that their payoffs change in the same direction.

As our main result, we considerably improve the characterization of the Shapley value by van den Brink (2001) and Casajus (2011). For games with more than two players, we show that the Shapley value can be characterized by efficiency, the null player property, and weak differential marginality (Theorem 2). Moreover, we provide a counterexample for games with two players (Appendix B).

The remainder of this paper is organized as follows. In Section 2, we give basic definitions and notation. In Section 3, we present our main result. Some remarks conclude this paper. An appendix contains the proof of our main result and some complementary findings.

### 2. Basic definitions and notation

A (finite TU) game on a non-empty and finite set of players \( N \) is given by a coalition function \( v \in V (N) := \{ f : 2^N \to \mathbb{R} \mid f(\emptyset) = 0 \} \), where \( 2^N \) denotes the power set of \( N \). Subsets of \( N \) are called coalitions; \( v(S) \) is called the worth of coalition \( S \). Since we deal

\(^{2}\) Ju and Wettstein (2009) suggest a class of bidding mechanisms that implement several solution concepts for TU games including the Shapley value.

\(^{3}\) As already mentioned by Young (1985), strong monotonicity can be relaxed into marginality, i.e., a player’s payoff only depends on her own productivity.

\(^{4}\) Casajus (2011, Proposition 4) shows that differential marginality coincides with fairness (van den Brink, 2001) on the full domain of games, for example.
with a fixed player set \( N \), the latter mostly is dropped as an argument. For \( v, w \in \mathbb{V}, \alpha \in \mathbb{R} \), the coalition functions \( v + w \in \mathbb{V} \) and \( \alpha \cdot v \in \mathbb{V} \) are given by \((v + w)(S) = v(S) + w(S)\) and \((\alpha \cdot v)(S) = \alpha \cdot v(S)\) for all \( S \subseteq N \). The game \( \Theta \in \mathbb{V} \) given by \( \Theta(S) = 0 \) for all \( S \subseteq N \) is called the **null game**. For \( T \subseteq N, T \neq \emptyset \), the game \( u_T \in \mathbb{V}, u_T(S) = 1 \) if \( T \subseteq S \) and \( u_T(S) = 0 \) otherwise, is called a **unanimity game**. Any \( v \in \mathbb{V} \) can be uniquely represented by unanimity games, i.e.,

\[
v = \sum_{T \subseteq N : T \neq \emptyset} \lambda_T(v) \cdot u_T,
\]

where the coefficients \( \lambda_T(v) \) can be determined recursively via

\[
v(S) = \sum_{T \subseteq S : T \neq \emptyset} \lambda_T(v) \quad \text{for all } S \subseteq N.
\]

Player \( i \in N \) is called a **dummy player** in \( v \in \mathbb{V} \) if \( v(S \cup \{i\}) - v(S) = v(\{i\}) \) for all \( S \subseteq N \setminus \{i\} \); player \( i \in N \) is called a **null player** in \( v \in \mathbb{V} \) if \( v(S \cup \{i\}) = v(S) \) for all \( S \subseteq N \setminus \{i\} \); players \( i, j \in N \) are called symmetric in \( v \in \mathbb{V} \) if \( v(S \cup \{i\}) = v(S \cup \{j\}) \) for all \( S \subseteq N \setminus \{i, j\} \).

A **value** on \( N \) is a mapping \( \varphi : \mathbb{V} \to \mathbb{R}^N \). The **Shapley value** (Shapley, 1953), \( \text{Sh} \), is given by

\[
\text{Sh}_i(v) := \sum_{T \subseteq N : i \in T} |T|^{-1} \cdot \lambda_T(v) \quad \text{for all } v \in \mathbb{V} \text{ and } i \in N.
\]

### 3. Weak differential marginality and the Shapley value

The Shapley value satisfies a very natural fairness condition due to van den Brink (2001). **Fairness, F.** For all \( v, w \in \mathbb{V} \) and \( i, j \in N \) such that \( i \) and \( j \) are symmetric in \( w \), we have

\[
\varphi_i(v + w) - \varphi_i(v) = \varphi_j(v + w) - \varphi_j(v).
\]

Fairness guarantees that if a game changes by adding another game in which two players are symmetric, then both players’ payoffs change by the same amount. Since adding such a game changes these players’ productivities by the same amount, this property is a rather natural requirement. Equal productivity differentials should translate into equal payoff differentials. Casajus (2011) states this idea directly with his differential marginality axiom, which is equivalent to the fairness property on the full domain of games (his Proposition 4).\(^5\)

**Differential marginality, DM.** For all \( v, w \in \mathbb{V} \) and \( i, j \in N \) such that

\[
v(S \cup \{i\}) - v(S \cup \{j\}) = w(S \cup \{i\}) - w(S \cup \{j\}) \quad \text{for all } S \subseteq N \setminus \{i, j\},
\]

we have

\[
\varphi_i(v) - \varphi_j(v) = \varphi_i(w) - \varphi_j(w).
\]

\(^5\)Casajus and Huettner (2013) consider a strengthened version of differential marginality called strong differential monotonicity: For all \( v, w \in \mathbb{V} \) and \( i, j \in N \) such that \( v(S \cup \{i\}) - v(S \cup \{j\}) \geq w(S \cup \{i\}) - w(S \cup \{j\}) \) for all \( S \subseteq N \setminus \{i, j\} \), we have \( \varphi_i(v) - \varphi_j(v) \geq \varphi_i(w) - \varphi_j(w) \).
Together with two standard properties, efficiency and the null player property, these axioms characterize the Shapley value.

**Efficiency, E.** For all $v \in \mathcal{V}$, we have $\sum_{\ell \in N} \varphi_{\ell} (v) = v (N)$.

**Null player, N.** For all $v \in \mathcal{V}$ and $i \in N$ such that $i$ is a null player in $v$, we have $\varphi_i (v) = 0$.

**Theorem 1** (van den Brink, 2001; Casajus, 2011). The Shapley value is the unique solution that satisfies efficiency ($E$), the null player property ($N$), and fairness ($F$)/differential marginality ($DM$).

In the following, we suggest a substantial relaxation of differential marginality. The implication of differential marginality can be rewritten as

$$\varphi_i (v) - \varphi_i (w) = \varphi_j (v) - \varphi_j (w).$$

Further, the hypothesis of differential marginality is satisfied if and only if players $i$ and $j$ are symmetric in $v - w$. Hence, differential marginality can be phrased as that equal changes in two players’ productivities should translate into equal changes of their payoffs. Of course, this implies that their payoffs change in the same direction, i.e.,

$$\varphi_i (v) \geq \varphi_i (w) \quad \text{if and only if} \quad \varphi_j (v) \geq \varphi_j (w).$$

Therefore, differential marginality implies the following considerably weaker property.

**Weak differential marginality, $DM^-$.** For all $v, w \in \mathcal{V}$ and $i, j \in N$ such that

$$v (S \cup \{i\}) - v (S \cup \{j\}) = w (S \cup \{i\}) - w (S \cup \{j\}) \quad \text{for all } S \subseteq N \setminus \{i, j\},$$

we have

$$\varphi_i (v) \geq \varphi_i (w) \quad \text{if and only if} \quad \varphi_j (v) \geq \varphi_j (w).$$

As our main result, we show that one can replace differential marginality with weak differential marginality in Theorem 1 for $|N| \neq 2$.

**Theorem 2.** Let $|N| \neq 2$. The Shapley value is the unique value that satisfies efficiency ($E$), the null player property ($N$), and weak differential marginality ($DM^-$).

The proof of Theorem 2 can be found in Appendix A. Appendix B contains the counterexample to our characterization for $|N| = 2$. The non-redundancy of our characterization for $|N| > 2$ is indicated in Appendix C.

If the null player property is strengthened into the dummy player property in Theorem 2, then it also holds for $|N| = 2$. It is straightforward to show this using the techniques employed for establishing the induction basis within the proof of Theorem 2.

**Dummy player, D.** For all $v \in \mathcal{V}$ and $i \in N$ such that $i$ is a dummy player in $v$, we have $\varphi_i (v) = v (\{i\})$.
4. Concluding remarks

Although differential marginality or differential monotonicity can be used to characterize the Shapley value, these properties show their full potential when it comes to solutions that fail marginality or strong monotonicity. Consider, for example, the egalitarian Shapley values (Joosten, 1996), which are the convex mixtures of the Shapley value and the equal division value. For games with more than two players, Casajus and Huettner (2013) characterize this class via efficiency, strong differential monotonicity, the null player in a productive environment property, where the latter requires a null player to obtain a non-negative payoff whenever the worth generated by the grand coalition is non-negative. In view of our main result, it seems to be interesting to explore whether in this characterization strong differential monotonicity can be relaxed in the same vein as weak differential monotonicity relaxes differential monotonicity. For example, van den Brink et al. (2013) and Casajus and Huettner (2014) use weak monotonicity, which is a relaxation of strong monotonicity, in order to characterize the class of egalitarian Shapley values.

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Appendix A. Proof of Theorem 2

Proof. Existence: It is well-known that $\text{Sh}$ satisfies $E$ and $N$. Since $DM$ implies $DM^-$, Casajus (2011, Corollary 5) entails that $\text{Sh}$ obeys $DM^-$. Uniqueness: Let the solution $\varphi$ meet $E$, $N$, and $DM^-$. If $|N| = 1$, then $E$ already implies $\varphi = \text{Sh}$. Let now $|N| > 2$. For $v \in \mathcal{V}$, set

$$T > 1 (v) := \{ T \subseteq N \mid |T| > 1 \text{ and } \lambda_T (v) \neq 0 \}.$$  

We show $\varphi = \text{Sh}$ by induction on $|T > 1 (v)|$. For $T \in T > 1 (v)$, let $v_T \in \mathcal{V}$ be given by

$$v_T := v - \lambda_T (v) \cdot u_T + \frac{\lambda_T (v)}{|T|} \cdot \sum_{\ell \in T} u(\ell).$$  \hspace{1cm} (A.1)

By construction, (*) $|T > 1 (v_T)| = |T > 1 (v)| - 1$ and (**) $v (N) = v_T (N)$. Further, for $T \subseteq N$, $|T| > 1$, let $\tilde{u}_T \in \mathcal{V}$ be given by

$$\tilde{u}_T := |T| \cdot u_T - \sum_{\ell \in T} u(\ell).$$  \hspace{1cm} (A.2)

Note that $\text{Sh}_i (\tilde{u}_T) = 0$ for all $i \in N$. 

**Induction basis:** Let \( v \in \mathbb{V} \) be such that \( |T_{>1}(v)| \leq 1 \). There are \( \alpha^v \in \mathbb{R}^N \) and \( \beta^v \in \mathbb{R} \), and \( T^v \subseteq N, |T^v| > 1 \) such that

\[
 v = \beta^v \cdot \bar{u}_{T^v} + \sum_{i \in N} \alpha_i^v \cdot u_{\{i\}}.
\]

Set \( R^v := \{i \in N \mid \alpha_i^v \neq 0\} \). We show that \( \varphi(v) = \text{Sh}(v) \) for all \( v \in \mathbb{V} \) with \( |T_{>1}(v)| \leq 1 \) by a number of claims.

**Claim 1, C1:** If \( \beta^v = 0 \) and \( R^v \neq N \), then \( \varphi(v) = \text{Sh}(v) \).

There exists some \( k \in N \setminus R^v \). By \( \textbf{E} \) and \( \textbf{N} \), we have

\[
 \varphi_k(v) = 0 = \varphi_k(\alpha_i^v \cdot u_{\{i\}}) = \text{Sh}_k(v) \quad \text{for all } i \in N \setminus \{k\} \tag{A.3}
\]

and

\[
 \varphi_i(\alpha_i^v \cdot u_{\{i\}}) = \text{Sh}_i(v) \quad \text{for all } i \in N. \tag{A.4}
\]

By (A.3) and \( \text{DM}^- \) applied to \( i \) and \( k \), we have

\[
 \varphi_i(v) = \varphi_i(\alpha_i^v \cdot u_{\{i\}}) \quad \text{for all } i \in N \setminus \{k\}. \tag{A.5}
\]

Now, the claim drops from (A.3), (A.4), and (A.5).

**Claim 2, C2:** If \( \beta^v = 0 \), then \( \varphi(v) = \text{Sh}(v) \).

Suppose \( \varphi(v) \neq \text{Sh}(v) \). By \( \textbf{E} \), there are \( i, j \in N \) such that

\[
 \varphi_i(v) > \text{Sh}_i(v) \quad \text{and} \quad \varphi_j(v) < \text{Sh}_j(v). \tag{A.6}
\]

Let \( k \in N \setminus \{i, j\} \). By \( \text{DM}^- \) and since \( i \) and \( j \) are symmetric in \( -\alpha_k \cdot u_{\{k\}} \), we have

\[
 \varphi_i(v) \gtrsim \varphi_i(v - \alpha_k \cdot u_{\{k\}}) \overset{\text{C1}}{=} \text{Sh}_i(v - \alpha_k \cdot u_{\{k\}}) = \text{Sh}_i(v)
\]

if and only if

\[
 \varphi_j(v) \gtrsim \varphi_j(v - \alpha_k \cdot u_{\{k\}}) \overset{\text{C1}}{=} \text{Sh}_j(v - \alpha_k \cdot u_{\{k\}}) = \text{Sh}_j(v),
\]

which contradicts (A.6).

**Claim 3, C3:** If \( |R^v| = 0 \), then \( \varphi(v) = \text{Sh}(v) \).

Note that \( v = \beta^v \cdot \bar{u}_{T^v} \). By \( \textbf{N} \), we have

\[
 \varphi_i(v) = 0 = \text{Sh}_i(v) \quad \text{for all } i \in N \setminus T^v. \tag{A.7}
\]

Since all players in \( T^v \) are pairwise symmetric in \( v \) and by \( \textbf{N} \) and \( \text{DM}^- \), we thus have

\[
 \varphi_i(v) \gtrsim \varphi_i(0) = 0 \quad \text{if and only if} \quad \varphi_j(v) \gtrsim \varphi_j(0) = 0 \quad \text{for all } i, j \in T^v. \tag{A.8}
\]

By \( \beta^v \cdot \bar{u}_{T^v}(N) = 0, \textbf{E}, \text{(A.7)}, \) and (A.8), we have \( \varphi_i(v) = 0 = \text{Sh}_i(v) \) for all \( i \in T^v \).
Claim 4, C4: If $R^v \cup T^v \neq N$, then $\varphi(v) = \text{Sh}(v)$.

We prove the claim by induction on $|R^v|$.

Induction basis C4: For $|R^v| = 0$, the claim holds by C3.

Induction hypothesis C4: Let the claim hold for $|R^v| \leq t$.

Induction step C4: Let $|R^v| = t + 1$. Suppose $\varphi(v) \neq \text{Sh}(v)$. By E, there are $i, j \in N$ such that

$$\varphi_i(v) > \text{Sh}_i(v) \quad \text{and} \quad \varphi_j(v) < \text{Sh}_j(v). \quad (A.9)$$

By N, we have $i, j \in R^v \cup T^v$.

Case (i): Suppose $i, j \in R^v \setminus T^v$ or $i, j \in T^v$. By C2, we have

$$\varphi(v - \beta^v \cdot \bar{u}_T) = \text{Sh}(v - \beta^v \cdot \bar{u}_T). \quad (A.10)$$

By (A.9) and (A.10), we further have

$$\varphi_i(v) - \varphi_i(v - \beta^v \cdot \bar{u}_T) > \text{Sh}_i(v) - \text{Sh}_i(v - \beta^v \cdot \bar{u}_T) = \text{Sh}_i(\beta^v \cdot \bar{u}_T) = 0$$

and

$$\varphi_j(v) - \varphi_j(v - \beta^v \cdot \bar{u}_T) < \text{Sh}_j(v) - \text{Sh}_j(v - \beta^v \cdot \bar{u}_T) = \text{Sh}_j(\beta^v \cdot \bar{u}_T) = 0.$$

Since $i$ and $j$ are symmetric in $-\beta^v \cdot \bar{u}_T$, this contradicts DM$^\rightarrow$.

Case (ii): Suppose, w.l.o.g., $i \in R^v \setminus T^v$ and $j \in T^v$. By Induction hypothesis C4, we have

$$\varphi_i(v - \alpha_i^v \cdot u_{(i)}) = \text{Sh}_i(v - \alpha_i^v \cdot u_{(i)}). \quad (A.11)$$

By assumption, there exists $k \in N \setminus (R^v \cup T^v)$. By (A.9) and (A.11), we have

$$\varphi_j(v) - \varphi_j(v - \alpha_i^v \cdot u_{(i)}) < \text{Sh}_j(v) - \text{Sh}_j(v - \alpha_i^v \cdot u_{(i)}) = \text{Sh}_j(\alpha_i^v \cdot u_{(i)}) = 0. \quad (A.12)$$

By N, we have

$$\varphi_k(v) = 0 = \text{Sh}_{k}(v - \alpha_i^v \cdot u_{(i)}). \quad (A.13)$$

Since $j$ and $k$ are symmetric in $-\alpha_i^v \cdot u_{(i)}$, (A.12) and (A.13) contradict DM$^\rightarrow$.

Claim 5, C5: If $|T^v \setminus R^v| \geq 1$, then $\varphi(v) = \text{Sh}(v)$.

Suppose $\varphi(v) \neq \text{Sh}(v)$. By E, there are $i, j \in N$ such that

$$\varphi_i(v) > \text{Sh}_i(v) \quad \text{and} \quad \varphi_j(v) < \text{Sh}_j(v). \quad (A.14)$$

By N, we have $i, j \in R^v \cup T^v$.

Case (i): Suppose $i, j \in R^v \setminus T^v$ or $i, j \in T^v$. By C2, we have

$$\varphi(v - \beta^v \cdot \bar{u}_T) = \text{Sh}(v - \beta^v \cdot \bar{u}_T). \quad (A.15)$$

By (A.14) and (A.15), we further have

$$\varphi_i(v) - \varphi_i(v - \beta^v \cdot \bar{u}_T) > \text{Sh}_i(v) - \text{Sh}_i(v - \beta^v \cdot \bar{u}_T) = \text{Sh}_i(\beta^v \cdot \bar{u}_T) = 0$$

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and

\[ \varphi_j(v) - \varphi_j(v - \beta^v \cdot \bar{u}_T^v) < \text{Sh}_j(v) - \text{Sh}_j(v - \beta^v \cdot \bar{u}_T^v) = \text{Sh}_j(\beta^v \cdot \bar{u}_T^v) = 0. \]

Since \( i \) and \( j \) are symmetric in \(-\beta^v \cdot \bar{u}_T^v\), this contradicts \( \text{DM}^- \).

**Case (ii):** Suppose, w.l.o.g., \( i \in T^v \setminus T^w \) and \( j \in T^w \).

**Case (ii-a):** Suppose \( j \in T^w \setminus T^v \). Let \( w := v - \beta^v \cdot \bar{u}_T^v - \beta^v \cdot \bar{u}(T^v \setminus \{j\}) \cup \{i\} \). By \( \text{C4} \), we have

\[ \varphi(w) = \text{Sh}(w). \tag{A.16} \]

By (A.14) and (A.16), we further have

\[ \varphi_i(v) - \varphi_i(w) > \text{Sh}_i(v) - \text{Sh}_i(w) = \text{Sh}_i(\beta^v \cdot \bar{u}_T^v + \beta^v \cdot \bar{u}(T^v \setminus \{j\}) \cup \{i\}) = 0 \]

and

\[ \varphi_j(v) - \varphi_j(w) < \text{Sh}_j(v) - \text{Sh}_j(w) = \text{Sh}_j(\beta^v \cdot \bar{u}_T^v + \beta^v \cdot \bar{u}(T^v \setminus \{j\}) \cup \{i\}) = 0. \]

Since \( i \) and \( j \) are symmetric in \(-\beta^v \cdot \bar{u}_T^v\), this contradicts \( \text{DM}^- \).

**Case (ii-b):** Suppose \( j \in T^v \cap T^w \). By assumption, there exists \( k \in T^v \setminus T^w \) such that \( k \neq i \) and \( k \neq j \). By \( \text{C2} \), we have

\[ \varphi_j(v - \beta^v \cdot \bar{u}_T^v) = \text{Sh}_j(v - \beta^v \cdot \bar{u}_T^v) \quad \text{and} \quad \varphi_k(v - \beta^v \cdot \bar{u}_T^v) = \text{Sh}_k(v - \beta^v \cdot \bar{u}_T^v). \tag{A.17} \]

Since \( j \) and \( k \) are symmetric in \(-\beta^v \cdot \bar{u}_T^v\), by (A.14) and \( \text{C2} \), we have

\[ \varphi_j(v) - \varphi_j(v - \beta^v \cdot \bar{u}_T^v) < \text{Sh}_j(v) - \text{Sh}_j(v - \beta^v \cdot \bar{u}_T^v) = \text{Sh}_j(\beta^v \cdot \bar{u}_T^v) = 0. \]

By \( \text{DM}^- \), we further have

\[ \varphi_k(v) - \varphi_k(v - \beta^v \cdot \bar{u}_T^v) < 0. \]

Since

\[ \varphi_k(v - \beta^v \cdot \bar{u}_T^v) \overset{(A.17)}{=} \text{Sh}_k(v - \beta^v \cdot \bar{u}_T^v) = \text{Sh}_k(v), \]

we obtain

\[ \varphi_k(v) < \text{Sh}_k(v). \tag{A.18} \]

Let \( z = v - \beta^v \cdot \bar{u}_T^v - \beta^v \cdot \bar{u}(T^v \setminus \{k\}) \cup \{i\} \). By (A.14), (A.18), and \( \text{C4} \), we have

\[ \varphi_i(v) - \varphi_i(z) > \text{Sh}_i(v) - \text{Sh}_i(z) = \text{Sh}_i(\beta^v \cdot \bar{u}_T^v + \beta^v \cdot \bar{u}(T^v \setminus \{k\}) \cup \{i\}) = 0 \tag{A.19} \]

and

\[ \varphi_k(v) - \varphi_k(z) < \text{Sh}_k(v) - \text{Sh}_k(z) = \text{Sh}_k(\beta^v \cdot \bar{u}_T^v + \beta^v \cdot \bar{u}(T^v \setminus \{k\}) \cup \{i\}) = 0. \tag{A.20} \]

Since \( i \in T^v \setminus T^w \) and \( k \in T^v \setminus T^w \), \( i \) and \( k \) are symmetric in \(-\beta^v \cdot \bar{u}_T^v \). \( \text{DM}^- \) hence, (A.19) and (A.20) contradict \( \text{DM}^- \).

**Claim 6, C6:** If \( R^v \cup T^v = N \), then \( \varphi(v) = \text{Sh}(v) \).
If \(|T^v \setminus R^v| \geq 1\), then the claim drops from \(C_5\). Let now \(|T^v \setminus R^v| = 0\). By assumption, we have \(R^v = N\). Suppose \(\varphi (v) \neq \text{Sh} (v)\). By \(E\), there are \(i, j \in N\) such that

\[
\varphi_i (v) > \text{Sh}_i (v) \quad \text{and} \quad \varphi_j (v) < \text{Sh}_j (v).
\]

Let \(k \in N \setminus \{i, j\}\). If \(k \in N \setminus T^v\), then \(C_4\) implies

\[
\varphi (v - \alpha_k \cdot u_{\{k\}}) = \text{Sh} (v - \alpha_k \cdot u_{\{k\}}).
\]

If \(k \in T^v\), then \(C_5\) implies \((A.21)\). By \((A.21)\) and \((A.22)\), we have

\[
\varphi_i (v) - \varphi_i (v - \alpha_k \cdot u_{\{k\}}) > \text{Sh}_i (v) - \text{Sh}_i (v - \alpha_k \cdot u_{\{k\}}) = \text{Sh}_i (\alpha_k \cdot u_{\{k\}}) = 0
\]

and

\[
\varphi_j (v) - \varphi_j (v - \alpha_k \cdot u_{\{k\}}) < \text{Sh}_j (v) - \text{Sh}_j (v - \alpha_k \cdot u_{\{k\}}) = \text{Sh}_j (\alpha_k \cdot u_{\{k\}}) = 0.
\]

Since \(i\) and \(j\) are symmetric in \(-\alpha_k \cdot u_{\{k\}}\), this contradicts \(DM^-\).

Note that the induction basis is proved by \(C_4\) and \(C_6\).

**Induction hypothesis:** Let the claim hold for all \(v \in V\) such that \(|T_{>1} (v)| \leq t\), \(t \in \mathbb{N}\), \(t \geq 1\).

**Induction step:** Let now \(v \in V\) be such that \(|T_{>1} (v)| = t + 1\). There exist \(S, T \in T_{>1} (v)\) such that \(S \neq T\). By \((3)\), \((A.1)\), \((*)\), and the induction hypothesis, we have

\[
\varphi (v_S) = \text{Sh} (v_S) = \text{Sh} (v_T) = \varphi (v_T).
\]

**Case (i):** \(S \cap T \neq \emptyset\). W.l.o.g., \(S \setminus T \neq \emptyset\). Let \(i \in S \cap T\) and \(j \in S \setminus T\). By \((A.23)\) and \(DM^-\), we have

\[
\varphi_\ell (v) \geq \text{Sh}_\ell (v) \quad \text{if and only if} \quad \varphi_i (v) \geq \text{Sh}_i (v) \quad \text{for all} \ \ell \in S,
\]

\[
\varphi_\ell (v) \geq \text{Sh}_\ell (v) \quad \text{if and only if} \quad \varphi_i (v) \geq \text{Sh}_i (v) \quad \text{for all} \ \ell \in T,
\]

\[
\varphi_\ell (v) \geq \text{Sh}_\ell (v) \quad \text{if and only if} \quad \varphi_j (v) \geq \text{Sh}_j (v) \quad \text{for all} \ \ell \in N \setminus T,
\]

and therefore

\[
\varphi_\ell (v) \geq \text{Sh}_\ell (v) \quad \text{if and only if} \quad \varphi_i (v) \geq \text{Sh}_i (v) \quad \text{for all} \ \ell \in N.
\]

**Case (ii):** \(S \cup T \neq \emptyset\). W.l.o.g., \(S \setminus T \neq \emptyset\). Let \(i \in N \setminus (S \cup T)\) and \(j \in S \setminus T\). By \((A.23)\) and \(DM^-\), we have

\[
\varphi_\ell (v) \geq \text{Sh}_\ell (v) \quad \text{if and only if} \quad \varphi_i (v) \geq \text{Sh}_i (v) \quad \text{for all} \ \ell \in N \setminus S,
\]

\[
\varphi_\ell (v) \geq \text{Sh}_\ell (v) \quad \text{if and only if} \quad \varphi_i (v) \geq \text{Sh}_i (v) \quad \text{for all} \ \ell \in N \setminus T,
\]

\[
\varphi_\ell (v) \geq \text{Sh}_\ell (v) \quad \text{if and only if} \quad \varphi_j (v) \geq \text{Sh}_j (v) \quad \text{for all} \ \ell \in S,
\]

and therefore

\[
\varphi_\ell (v) \geq \text{Sh}_\ell (v) \quad \text{if and only if} \quad \varphi_i (v) \geq \text{Sh}_i (v) \quad \text{for all} \ \ell \in N.
\]
Case (iii): \( S \cap T = \emptyset \) and \( S \cup T = N \). Hence, \( T_{-1}(v) = \{S, T\} \). Let \( i \in S \), \( j \in T \), and \( w \in \mathbb{V} \) be given by

\[
w = v_S - \lambda_S(v) \cdot u_{(S \setminus \{i\}) \cup \{j\}} + \frac{\lambda_S(v)}{|S|} \sum_{\ell \in (S \setminus \{i\}) \cup \{j\}} u_{(S \setminus \{i\}) \cup \{j\}}. \tag{A.26}
\]

By construction, we have \( T_{-1}(w) = \{(S \setminus \{i\}) \cup \{j\}, T\} \) and \( \varphi(v)(N) = w(N) \). In view of Case (i), we have \( \varphi(w) = \text{Sh}(w) \).

Since \( i \) and \( j \) are symmetric in \( v - w \) and by \( \text{DM}^- \) and \( \text{(A.23)} \), we have

\[
\begin{align*}
\varphi_j(v) & \geq \varphi_j(w) \quad \text{(****)} \quad \text{Sh}_j(v) \quad \text{if and only if} \quad \varphi_i(v) \geq \varphi_i(w) \quad \text{(****)} \quad \text{Sh}_i(v) , \\
\varphi_{\ell}(v) & \geq \text{Sh}_{\ell}(v) \quad \text{if and only if} \quad \varphi_i(v) \geq \text{Sh}_i(v) \quad \text{for all } \ell \in S, \\
\varphi_{\ell}(v) & \geq \text{Sh}_{\ell}(v) \quad \text{if and only if} \quad \varphi_j(v) \geq \text{Sh}_j(v) \quad \text{for all } \ell \in T,
\end{align*}
\]

and therefore

\[
\varphi_{\ell}(v) \geq \text{Sh}_{\ell}(v) \quad \text{if and only if} \quad \varphi_i(v) \geq \text{Sh}_i(v) \quad \text{for all } \ell \in N. \tag{A.27}
\]

Finally, \( \text{(A.24)}, \text{(A.25)}, \text{(A.27)}, \text{and E} \) imply \( \varphi(v) = \text{Sh}(v) \).

\section*{Appendix B. Counterexample to Theorem 2 for \(|N| = 2\)}

Theorem 2 fails for \(|N| = 2\). Let \( N = \{1, 2\} \). Consider the solution \( \varphi^\triangledown : \mathbb{V} \to \mathbb{R}^2 \) given by

\[
(\varphi_1^\triangledown(v), \varphi_2^\triangledown(v)) = \begin{cases}
(\text{Sh}_1(v), \text{Sh}_2(v)), & \text{Sh}_1(v) \geq 0, \text{Sh}_2(v) \geq 0, \\
\left(\frac{\text{Sh}_1(v)}{2}, \text{Sh}_2(v) + \frac{\text{Sh}_1(v)}{2}\right), & \text{Sh}_1(v) > 0, \text{Sh}_2(v) < 0, v(N) \geq 0, \\
\left(\frac{\text{Sh}_1(v)}{2}, \text{Sh}_2(v) + \frac{\text{Sh}_1(v)}{2}\right), & \text{Sh}_1(v) > 0, \text{Sh}_2(v) < 0, v(N) < 0, \\
\left(\text{Sh}_1(v), \frac{\text{Sh}_2(v)}{2}, \frac{\text{Sh}_1(v)}{2}\right), & \text{Sh}_1(v) \leq 0 \text{Sh}_2(v) \leq 0, \\
\left(\text{Sh}_1(v) + \frac{\text{Sh}_2(v)}{2}, \frac{\text{Sh}_1(v)}{2}\right), & \text{Sh}_1(v) < 0, \text{Sh}_2(v) > 0, v(N) \geq 0, \\
\left(\frac{\text{Sh}_1(v)}{2}, \text{Sh}_2(v) + \frac{\text{Sh}_1(v)}{2}\right), & \text{Sh}_1(v) < 0, \text{Sh}_2(v) > 0, v(N) < 0
\end{cases}
\]

for all \( v \in \mathbb{V} \). One easily checks that \( \varphi^\triangledown \neq \text{Sh} \) and that \( \varphi^\triangledown \) inherits \( \text{E}, \text{N}, \) and \( \text{DM}^- \) from \text{Sh}.

\section*{Appendix C. Non-redundancy of Theorem 2 for \(|N| > 2\)}

Our characterization is non-redundant for \(|N| > 2\). The value \( \varphi^E \) given by \( \varphi^E_i(v) = 0 \) for all \( v \in \mathbb{V} \) and \( i \in N \) satisfies \( \text{N} \) and \( \text{DM}^- \) but not \( \text{E} \). The value \( \varphi^N \) given by \( \varphi^N_i(v) = 0 \) for all \( i \in N \) satisfies \( \text{E} \) and \( \text{DM}^- \) but not \( \text{N} \).
\( v(N)/|N| \) for all \( v \in V \) and \( i \in N \) satisfies \( E \) and \( \text{Mo}^- \) but not \( \text{N} \). For \( v \in V \), let \( N_0(v) := \{ i \in N \mid i \text{ is a null player in } v \} \). The value \( \varphi^{\text{DM}^-} \) given by

\[
\varphi^{\text{DM}^-}_i(v) = \begin{cases} 
\frac{v(N)}{|N \setminus N_0(v)|}, & i \in N \setminus N_0(v), \\
0, & i \in N_0(v)
\end{cases}
\]

for all \( v \in V \) and \( i \in N \)

satisfies \( E \) and \( \text{N} \) but not \( \text{DM}^- \).

References


