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Asymptotic stability in the dual Lovász-Shapley and the Shapley² replicator dynamics for TU games

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Casajus, Kramm, and Wiese (2020, J. Econ. Theory 186, 104993) study the asymptotic stability in population dynamics derived from finite cooperative games with transferable utility using the Lovász-Shapley value (Casajus and Wiese, 2017, Int. J. Game Theory 45, 1-16) for non-negatively weighted games, where the players are interpreted as types of individuals. We extend their analysis to the population dynamics derived using the dual Lovász-Shapley value and the Shapley² value for non-negatively weighted games (Casajus and Kramm, HHL Working Paper 196, HHL Leipzig Graduate School of Management, Leipzig, Germany). As the former, we provide a complete description of asymptotically stable population profiles in both dynamics. In the dual Lovász-Shapley replicator dynamic, for example, any asymptotically stable population profile is characterized by a coalition: while the types in the coalition have the same positive share, the other types vanish. In the dual of the game, the per-capita productivity of such a stable coalition must be greater than the per-capita productivity of any proper sub- or supercoalition. In simple monotonic games, this means that exactly the minimal blocking coalitions are stable.

Asymptotic stability in the dual Lovász-Shapley and the Shapley^2 replicator dynamics for TU games

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Abstract

Casajus, Kramm, and Wiese (2020, J. Econ. Theory 186, 104993) study the asymptotic stability in population dynamics derived from finite cooperative games with transferable utility using the Lovász-Shapley value (Casajus and Wiese, 2017, Int. J. Game Theory 45, 1-16) for non-negatively weighted games, where the players are interpreted as types of individuals. We extend their analysis to the population dynamics derived using the dual Lovász-Shapley value and the Shapley^2 value for non-negatively weighted games (Casajus and Kramm, HHL Working Paper 196, HHL Leipzig Graduate School of Management, Leipzig, Germany). As the former, we provide a complete description of asymptotically stable population profiles in both dynamics. In the dual Lovász-Shapley replicator dynamic, for example, any asymptotically stable population profile is characterized by a coalition: while the types in the coalition have the same positive share, the other types vanish. In the dual of the game, the per-capita productivity of such a stable coalition must be greater than the per-capita productivity of any proper sub- or supercoalition. In simple monotonic games, this means that exactly the minimal blocking coalitions are stable.

Keywords: Evolutionary game theory, Lovász-Shapley value, dual Lovász-Shapley value, Shapley^2 value, CES production function
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1. Introduction

For a long time, non-cooperative game theory has employed the evolutionary approach to model the behavior of boundedly rational agents and to analyze the relation between the stable outcomes of the dynamical process generated by their repeated interaction and the common static solution concept of Nash equilibrium (see, e.g., Mailath, 1998; Samuelson,

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2002). In contrast to non-cooperative game theory, cooperative game theory shifts the focus from modeling strategic interactions and the agents' strategic reasoning to payoffs: it asks questions about which payoff distribution among the players of a given cooperative game can, for instance, be considered as fair or stable (see, e.g., Kannai, 1992; Winter, 2002).

Recently, Casajus et al. (2020) derive population dynamics from finite cooperative games with transferable utility (TU games), where the players are interpreted as types of individuals. In particular, they use the Lovász-Shapley value (Casajus and Wiese, 2017) for non-negatively weighted TU games, that is, TU games that are enriched with non-negative weights of the players. The latter are interpreted as the sizes of the subpopulations of individuals of a particular type. The types' payoffs for a population configuration are interpreted as their fitness, which determines the growth of their subpopulations in continuous time. This gives rise to replicator dynamics, in particular, those that describe the development of the types' subpopulation shares in time.

Casajus et al. (2020, Theorem 1) provide a complete description of the asymptotically stable population profiles in the Lovász-Shapley replicator dynamic. A population profile is asymptotically stable if it is supported by a coalition that is stable in the following sense. The per-capita worth generated by such a coalition is strictly greater than for its proper subcoalitions and its proper supercoalitions. Whereas types within this coalition have the same population share, types outside vanish. This result is driven by the particular technology embodied in the Lovász-Shapley value, which is based on the Lovász extension operator (Lovász, 1983; Algaba et al., 2004), which in turn is based on a particular constant elasticity of substitution (CES) production function, the minimum operator.

Later on, Casajus (2021) adds some justification for the use of Lovász extension operator in the construction of the Lovász-Shapley value and therefore the use of the latter to construct replicator dynamics from TU games. The Lovász extension operator is the unique extension operator in the sense of Algaba et al. (2004) that satisfies two economically sound properties, a carrier property and the monotonicity of extensions for monotonic games. Only two other CES technologies give rise to extension operators that economically sound in the afore-mentioned sense, the maximum operator and the average operator. Based on these operators, Casajus and Kramm (2021) first introduce the dual Lovász extension operator and the Shapley extension operator and then the dual Lovász-Shapley solution and the Shapley^2 solution for non-negatively weighted TU games Casajus and Kramm (2022).

In this paper, we study the relative replicator dynamics induced by the dual Lovász-Shapley solution and the Shapley^2 solution. In the dual Lovász-Shapley replicator dynamic, a population profile is asymptotically stable if and only it is supported by a coalition that is stable in the following sense. In the dual of the TU game, the per-capita worth generated by such a coalition is strictly greater than for its proper subcoalitions and its proper supercoalitions. Whereas types within this coalition have the same population share, types outside vanish (Theorem 6). In the Shapley^2 replicator dynamic, a population profile is asymptotically stable if and only if it is supported by a type with the uniquely highest payoff—all other types vanish (Theorem 13).

This paper is organized as follows. In the second section, we provide basic definitions and notation. In the third section, we establish the relation between non-negatively weighted solutions and population dynamics. In the fourth section, we provide a brief survey on the Lovász-Shapley replicator dynamic. In the fifth section, we introduce and discuss the dual Lovász-Shapley replicator dynamic. In the sixth section, we introduce and discuss the Shapley[2] replicator dynamic. Some remarks conclude the paper. The Appendix contains all proofs.

2. Basic definitions and notation

A **TU game** (or simply game) for a non-empty and finite player set N is given by a **coalition function** $v : 2^N \to \mathbb{R}$, $v(\emptyset) = 0$, where 2^N denotes the power set of N. Subsets of N are called **coalitions**; $v(S)$ is called the worth of coalition S. The set of all TU games for N is denoted by $\mathbb{V}$. In the following, we refer to the members of $\mathbb{V}$ as **games**.

For $v, w \in \mathbb{V}$, and $\alpha \in \mathbb{R}$, the games $v + w \in \mathbb{V}$ and $\alpha \cdot v \in \mathbb{V}$ are given by $(v + w)(S) = v(S) + w(S)$ and $(\alpha \cdot v)(S) = \alpha \cdot v(S)$ for all $S \subseteq N$. For $v \in \mathbb{V}$, its **dual** $v^* \in \mathbb{V}$ is given by $v^*(S) = v(N) - v(N \setminus S)$. For $T \subseteq N$, $T \neq \emptyset$, the game $u_T \in \mathbb{V}$ given by $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise is called a **unanimity game**.

A game $v \in \mathbb{V}$ is called **monotonic** if $v(S) \leq v(T)$ for all $S, T \subseteq N$ such that $S \subseteq T$; it is called superadditive $v(S) + v(T) \leq v(S \cup T)$ for all $S, T \subseteq N$ such that $S \cap T = \emptyset$. A game $v \in \mathbb{V}$ is called **simple** if $v(N) = 1$ and $v(S) \in \{0, 1\}$ for all $S \subseteq N$. A coalition $S \subseteq N$ is called **winning** in a simple game v if $v(S) = 1$. A coalition $S \subseteq N$ is called **minimal winning** in a simple monotonic game v if it is winning and and no proper subcoalition is winning. A coalition $S \subseteq N$ is called **blocking** in a simple game v if $v(N \setminus S) = 0$. A coalition $S \subseteq N$ is called **minimal blocking** in a simple monotonic game $v \in \mathbb{V}$ if it is blocking and and no proper subcoalition is blocking. Note that $S \subseteq N$ is (minimal) blocking in a simple monotonic game v if and only if S is (minimal) winning in v^*.

A **rank order** for N is a bijection $\rho : N \to \{1, 2, \ldots, |N|\}$ with the interpretation that i is the $\rho(i)$th player in ρ; the set of rank orders of N is denoted by R. The set of players before i in ρ is denoted by $B_i(\rho) = \{\ell \in N : \rho(\ell) < \rho(i)\}$. The **marginal contribution** of i in ρ and v is denoted by

$$MC_i^v(\rho) := v(B_i(\rho) \cup \{i\}) - v(B_i(\rho)). \tag{1}$$

A **solution** (value) for $\mathbb{V}$ is a mapping $\varphi : \mathbb{V} \to \mathbb{R}^N$ assigning a payoff $\varphi_i(v)$ to any player $i \in N$ in any game $v \in \mathbb{V}$. The **Shapley value** (Shapley, 1953), Sh, is given by

$$\mathrm{Sh}_i(v) := \sum_{\rho \in R} \frac{MC_i^v(\rho)}{|R|} \qquad \text{for all } v \in \mathbb{V} \text{ and } i \in N. \tag{2}$$

A **non-negatively weighted game** is a pair $(v, s) \in \mathbb{V} \times \mathbb{R}_+^N$. A **non-negatively weighted solution** is a mapping $\varphi : \mathbb{V} \times \mathbb{R}_+^N \to \mathbb{R}^N$. The **Lovász-Shapley value** (Casajus and Wiese, 2017), LS, is given by

$$\mathrm{LS}_i(v, s) := s_i \cdot \sum_{\rho \in R(s)} \frac{MC_i^v(\rho)}{|R(s)|} \qquad \text{for all } v \in \mathbb{V},\ s \in \mathbb{R}_+^N, \text{ and } i \in N, \tag{3}$$

where

$$R(s) := \{\rho \in R \mid \rho(i) < \rho(j) \text{ for all } i, j \in N \text{ with } s_i > s_j\}. \tag{4}$$

It is based on the **Lovász extension operator** (see Lovász, 1983; Algaba et al., 2004).

3. Non-negatively weighted solutions and population dynamics

Casajus et al. (2020) interpret the players in non-negatively weighted games as types of agents and the players' weights as the size of their respective subpopulations in the population of all agents. Moreover, they interpret the types' payoffs for non-negatively weighted solutions as the fitness of their subpopulations, which determines their growth in (continuous) time. For any non-negatively weighted solution φ and game $v \in \mathbb{V}$, this gives rise to the absolute population dynamic

$$\dot{s}_i = \frac{\partial s_i}{\partial t} = \varphi_i(v, s) \qquad \text{for all } i \in N \text{ and } s \in \mathbb{R}^N_+. \tag{5}$$

When the non-negatively weighted solution φ is positively homogenous in s, that is,

$$\varphi(v, \alpha \cdot s) = \alpha \cdot \varphi(v, s) \qquad \text{for all } \alpha \in \mathbb{R}_+,$$

one can pass directly to the relative population dynamic, that is, the dynamic of population profiles, $\left(x_i = s_i / \sum_{\ell \in N} s_\ell\right)_{i \in N}$,

$$\dot{x}_i = \varphi_i(v, x) - x_i \cdot \sum_{\ell \in N} \varphi_\ell(v, x) \tag{6}$$

for all $i \in N$ and $x \in \Delta^N_+$, where

$$\Delta^N_+ := \left\{x \in \mathbb{R}^N_+ \mid \textstyle\sum_{\ell \in N} x_\ell = 1\right\}.$$

Depending on the properties of the non-negatively weighted solution φ, one can apply one or another notion of a solution to the differential equations (5) and (6).

4. The Lovász-Shapley replicator dynamic[1]

Casajus et al. (2020) use the Lovász-Shapley value in (5) and (6) leading to the absolute population dynamic

$$\dot{s}_i = s_i \cdot \sum_{\rho \in R(s)} \frac{MC^v_i(\rho)}{|R(s)|} \qquad \text{for all } i \in N \text{ and } s \in \mathbb{R}^N_+. \tag{7}$$

[1] The survey in this section closely follows Casajus et al. (2020). For a more detailed discussion of this approach, we refer the reader to this paper.

Since the Lovász-Shapley value is positively homogenous in s, one can pass directly to the relative population dynamic

$$\dot{x}_i = g_i(x) := x_i \cdot \left(\sum_{\rho \in R(x)} \frac{MC_i^v(\rho)}{|R(x)|} - \sum_{\ell \in N} l \right) \tag{8}$$

for all $i \in N$ and $x \in \Delta_+^N$, where $g : \Delta_+^N \to \mathbb{R}^N$. Note that we actually obtain a *replicator* dynamic. In (7), for example, we obtain the fitness of type i by multiplying the fitness of single agent of type i, $\sum_{\rho \in R(s)} MC_i^v(\rho) / |R(s)|$, with its subpopulation size s_i. In the following, we restrict attention to the relative population dynamic.

By (3) and particularly (4), the right-hand side of (6) is discontinuous in general. Hence, one cannot apply the standard notion of a solution to them. For $\rho \in R$, let

$$\Delta_+^N(\rho) := \left\{ x \in \Delta_+^N \mid x_i > x_j \text{ for all } i, j \in N \text{ with } \rho(i) < \rho(j) \right\}.$$

Note that $R(x) = \{\rho\}$ for $x \in \Delta_+^N(\rho)$ and therefore

$$g_i(x) = x_i \cdot \left(MC_i^v(\rho) - \sum_{\ell \in N} x_\ell \cdot MC_\ell^v(\rho) \right) \qquad \text{for all } i \in N,\ \rho \in R(x),\ \text{and } x \in \Delta_+^N(\rho).$$

This implies that the function g is continuous up to its border and even Lipschitz continuous on $\Delta_+^N(\rho)$. The Filippov solution (Filippov, 1988) uses this insight. Set

$$C := \bigcup_{\rho \in R} \Delta_+^N(\rho) \qquad \text{and} \qquad D := \Delta_+^N \setminus C.$$

In order to handle the possible discontinuities of g on D, one constructs the differential inclusion

$$\begin{aligned} \dot{x} \in G(x) &:= \operatorname{con} \left\{ \lim_{(x^k) \in \Delta_+^N(\rho) : (x^k) \to x} g\left(x^k\right) \mid \rho \in \Delta_+^N(x) \right\} \\ &= \begin{cases} \{g(x)\}, & x \in C, \\ \operatorname{con} \left\{ \lim_{(x^k) \in \Delta_+^N(\rho) : (x^k) \to x} g\left(x^k\right) \mid \rho \in \Delta_+^N(x) \right\}, & x \in D \end{cases} \end{aligned} \tag{9}$$

for all $x \in \Delta_+^N$, where $G : \Delta_+^N \rightrightarrows \mathbb{R}^N$, con denotes the convex hull, and $\left(x^k\right) \to x$ means that the sequence $\left(x^k\right)$ converges to x.[2]

Definition 1 (Filippov solution). *A Filippov solution at $x^0 \in \Delta_+^N$ of the differential equation (8) with a discontinuous right-hand side is an absolutely continuous function $\xi : I \to \Delta_+^N$ for which $\xi(0) = x^0$ and with respect to the differential inclusion (9) it satisfies $\dot{\xi}(t) \in G(\xi(t))$ almost everywhere on some interval $I := [0, T] \subseteq \mathbb{R}$.*

[2] Note that there are typos in Casajus et al. (2020, Equation 11), which have been corrected in Equation 9.

Since Filippov solutions may not be unique, one needs to adapt the notion of asymptotic stability accordingly.

Definition 2 (Asymptotic stability). *A population profile $x^* \in \Delta_+^N$ is asymptotically stable in (7), if both of the following conditions hold: For any $\varepsilon > 0$, there exists some $\delta > 0$ such that for **any** solution ξ to (7) and any $x^0 \in \Delta_+^N$ such that $\|x^* - x^0\| < \delta$ and $\xi(0) = x^0$, we have*

1. $\|x^* - \xi(t)\| < \varepsilon$ *for all* $t \in [0, \infty)$ *and*
2. $\lim_{t\to\infty} \|x^* - \xi(t)\| = 0$.

The following theorem provides a complete description of the asymptotically stable population profiles for all games.

Theorem 3 (Casajus et al., 2020). *A population profile $x \in \Delta_+^N$ is asymptotically stable in the Lovász-Shapley replicator dynamic for $v \in \mathbb{V}$ if and only if there exists some $S \subseteq N$, $S \neq \emptyset$ such that $\frac{v(S)}{|S|} > \frac{v(T)}{|T|}$ for all $T \subseteq N$ such that $T \subsetneq S$ or $T \supsetneq S$ and $x = 1_S$, where $1_S \in \Delta_+^N$ is given by $(1_S)_i = |S|^{-1}$ for $i \in S$ and $(1_S)_i = 0$ for $i \in N \setminus S$.*

Asymptotic stability in the Lovász-Shapley replicator dynamic for a game v identifies coalitions S that are stable in the following sense. The per-capita worth generated by S is strictly greater than that of its proper subcoalitions and its proper supercoalitions. This also reflects the local nature of asymptotic stability. From this theorem, the following two corollaries are immediate.

Corollary 4 (Casajus et al., 2020). *In the Lovász-Shapley repicator dynamic, (i) an asymptotically stable population profile generically*[3] *exists, and*

(ii) asymptotically stable population profiles are robust to small perturbations of the underlying game.

The second part of the corollary says that small "mutations" in the (relative) productivity of the types within a population do not affect the asymptotically stable population profiles in the Lovász-Shapley replicator dynamic for a game.

Corollary 5 (Casajus et al., 2020). *In the Lovász-Shapley repicator dynamic, a population profile $x \in \Delta_+^N$ is asymptotically stable for a simple monotonic game $v \in \mathbb{V}$, if and only if there exists a minimal winning coalition S in v such that $x = 1_S$.*

This corollary says the Lovász-Shapley replicator dynamic essentially identifies minimal winning coalitions in simple monotonic games.

[3] We employ the following notion of genericity: A property of games holds generically if there exists an open and dense subset $\mathcal{V}$ of $\mathbb{V}$ such that all games in $\mathcal{V}$ show this property.

5. The dual Lovász-Shapley replicator dynamic

Casajus and Kramm (2022) motivate and advocate the **dual Lovász-Shapley solution** for positively weighted games, which is based on the dual Lovász extension operator (see Casajus and Kramm, 2021). It is given by

$$\mathrm{LS}_i^*(v,s) = |R^*(s)|^{-1} \cdot \sum_{\rho\in R^*(s)} s_i \cdot MC_i^v(\rho) \qquad \text{for all } v\in\mathbb{V},\ s\in\mathbb{R}_+^N, \text{ and } i\in N, \tag{10}$$

where

$$R^*(s) := \{\rho\in R \mid \rho(i) < \rho(j) \ \text{ for all } i,j\in N \text{ with } s_i < s_j\}. \tag{11}$$

It is related to the Lovász-Shapley solution by the following fact Casajus and Kramm (2022, Proposition 4). We have

$$\mathrm{LS}^*(v,s) = \mathrm{LS}(v^*,s) \qquad \text{for all } v\in\mathbb{V} \text{ and } s\in\mathbb{R}_+^N. \tag{12}$$

Using the dual Lovász-Shapley solution in (6), we obtain the (relative) dual Lovász-Shapley replicator dynamic

$$\dot{x}_i = g_i(x) := x_i \cdot \left(\sum_{\rho\in R^*(x)} \frac{MC_i^v(\rho)}{|R^*(x)|} - \sum_{\ell\in N} x_\ell \cdot \sum_{\rho\in R^*(x)} \frac{MC_\ell^v(\rho)}{|R^*(x)|} \right) \tag{13}$$

for all $i\in N$ and $x\in\Delta_+^N$, where $g:\Delta_+^N\to\mathbb{R}^N$. Note that the right-hand side of (13) has similar properties as the right-hand side of (8). Hence, we can apply the Filippov solution (Definition 1) and asymptotic stability (Definition 2). Using Theorem 3 and Equation (12), we obtain a complete description of the asymptotically stable population profiles for all games.

Theorem 6. *A population profile $x\in\Delta_+^N$ is asymptotically stable in the dual Lovász-Shapley replicator dynamic for $v\in\mathbb{V}$ if and only if there exists some $S\subseteq N$, $S\neq\emptyset$ such that $\frac{v^*(S)}{|S|} > \frac{v^*(T)}{|T|}$ for all $T\subseteq N$ such that $T\subsetneq S$ or $T\supsetneq S$ and $x=1_S$.*

A population profile is asymptotically stable in the dual Lovász-Shapley replicator dynamic for a game v if and only if it is supported by a coalition S that is stable in the following sense. The per-capita worth generated by S in the dual v^* of the game v is strictly greater than that of its proper subcoalitions and its proper supercoalitions. Whereas the types in $N\setminus S$ vanish, the types in S have the same population share. This also reflects the local nature of asymptotic stability. From this theorem, the following two corollaries are immediate.

Corollary 7. *In the dual Lovász-Shapley replicator dynamic, (i) an asymptotically stable population profile generically (see Footnote 3) exists, and*

(ii) asymptotically stable population profiles are robust to small perturbations of the underlying game.

The second part of the corollary says that small "mutations" in the (relative) productivity of the types within a population do not affect the asymptotically stable population profiles in the dual Lovász-Shapley replicator dynamic for a game.

Corollary 8. *In the dual Lovász-Shapley replicator dynamic, a population profile $x \in \Delta_+^N$ is asymptotically stable for a simple monotonic game $v \in \mathbb{V}$, if and only if there exists a minimal blocking coalition S in v such that $x = 1_S$.*

This corollary says the dual Lovász-Shapley replicator dynamic essentially identifies minimal blocking coalitions in simple monotonic games. Compare this with Lovász-Shapley replicator dynamic that identifies minimal winning coalitions (Corollary 5).

6. The Shapley² replicator dynamic

Casajus and Kramm (2022) motivate and advocate the **Shapley² solution** for positively weighted games, which is based on the Shapley extension operator (see Casajus and Kramm, 2021). It is given by

$$\mathrm{SS}_i(v,s) := s_i \cdot \mathrm{Sh}_i(v) \qquad \text{for all } v \in \mathbb{V} \text{ and } s \in \mathbb{R}_+^N. \tag{14}$$

Using the Shapley² solution in (6), we obtain the (relative) Shapley² replicator dynamic

$$\dot{x}_i = g_i(x) := x_i \cdot \left(\mathrm{Sh}_i(v) - \sum_{\ell \in N} x_\ell \cdot \mathrm{Sh}_i(v) \right) \tag{15}$$

for all $i \in N$ and $x \in \Delta_+^N$, where $g : \Delta_+^N \to \mathbb{R}^N$. In contrast to (8) and (13), the right-hand side of (15) is Lipschitz continuous. Hence, we can apply the Picard-Lindelöf theorem, which guaranties a unique solution for the differential equations (15), and then the standard notion of asymptotic stability.

Definition 9. *A solution at $x^0 \in \Delta_+^N$ of the differential equation (15) is a differentiable function $\xi : I \to \Delta_+^N$ for which $\xi(0) = x^0$ and with respect to the differential equation (15) it satisfies $\dot{\xi}(t) = g(\xi(t))$ for all $t \in I := [0,T] \subseteq \mathbb{R}$.*

Definition 10. *A population profile $x^* \in \Delta_+^N$ is asymptotically stable in (7), if both of the following conditions hold: For any $\varepsilon > 0$, there exists some $\delta > 0$ such that for the unique solution ξ to (7) and any $x^0 \in \Delta_+^N$ such that $\|x^* - x^0\| < \delta$ and $\xi(0) = x^0$, we have*

1. $\|x^* - \xi(t)\| < \varepsilon$ *for all* $t \in [0,\infty)$ *and*
2. $\lim_{t\to\infty} \|x^* - \xi(t)\| = 0$.

Equation (15) describes a system of non-linear, coupled first order ordinary differential equations. We use the insights from Lyapunov theory to analyze the stability of the system. Note that $x \in \Delta_+^N$ is an asymptotically stable point if and only if it is a strict local maximizer of a Lyapunov function. Thus, we introduce the following common definition.

Definition 11 (Lyapunov function). *The C^1-function $L : \mathbb{R}^N \to \mathbb{R}$ is an (increasing) strict Lyapunov function for the differential equation $\dot{x} = g(x)$ if $\dot{L}(x) \geq 0$ for all $x \in \Delta_+^N$, with equality only at stationary points of g.*

As it is commonly known, the definition implies that L increases along a solution and is positive semi-definite (see Sydsaeter et al., 2008, pp. 273).

Lemma 12. *A Lyapunov function of the dynamic system (15) is given by*

$$L(x) = \sum_{i \in N} x_i \cdot \mathrm{Sh}_i(v).$$

Proof. Using vector notation, due to the chain rule, we have

$$\dot{L}(x) = \nabla L(x) \cdot \dot{x} = \sum_{i \in N} \dot{x}_i \cdot \mathrm{Sh}_i(v)$$

Hence, we have $\dot{L}(x) = 0$ if $\dot{x} = 0$, that is, if x is a stationary point of g. Now, take some state x, where $\dot{x} \neq 0$. We can always divide the types into the two sets, $N_+(x) := \{i \in N \mid \dot{x}_i > 0\}$ and $N_-(x) := \{i \in N \mid \dot{x}_i < 0\}$ according to their growth rate in state x. (*) By (15), we have $\mathrm{Sh}_i(v) > \sum_{\ell \in N} x_\ell \cdot \mathrm{Sh}_\ell(v)$ for $i \in N_+(x)$ and $\mathrm{Sh}_i(v) < \sum_{\ell \in N} x_\ell \cdot \mathrm{Sh}_\ell(v)$ for $i \in N_-(x)$, and therefore $N_+(x) \neq \emptyset$ and $N_-(x) \neq \emptyset$. Rewriting the above expression yields

$$\begin{aligned}
\sum_{i \in N} \dot{x}_i \cdot \mathrm{Sh}_i(v) &= \sum_{i \in N_+(x)} \dot{x}_i \cdot \mathrm{Sh}_i(v) + \sum_{i \in N_-(x)} \dot{x}_i \cdot \mathrm{Sh}_i(v) \\
&\geq \left[\min_{i \in N_+(x)} \mathrm{Sh}_i(v)\right] \cdot \sum_{i \in N_+(x)} \dot{x}_i + \left[\max_{i \in N_-(x)} \mathrm{Sh}_i(v)\right] \cdot \sum_{i \in N_-(x)} \dot{x}_i \\
&= \underbrace{\left(\left[\min_{i \in N_+(x)} \mathrm{Sh}_i(v)\right] - \left[\max_{i \in N_-(x)} \mathrm{Sh}_i(v)\right]\right)}_{>0 \text{ by } (*)} \cdot \underbrace{\sum_{i \in N_+(x)} \dot{x}_i}_{>0} > 0
\end{aligned}$$

using $-\sum_{i \in N_+(x)} \dot{x}_i = \sum_{i \in N_-(x)} \dot{x}_i$, in the last step, which holds, as the population states have to lie in the simplex. This implies that L increases along all solutions to $\dot{x} = g(x)$ with the starting condition $x^0 \in \Delta_+^N$. □

In contrast to the Lyapunov function for the (relative) Lovász-Shapley replicator dynamic Casajus et al. (2020, Lemma 1), the Lyapunov function for the (relative) Shapley² replicator dynamic is affine on the full simplex. This immediately implies the following complete description of the asymptotically stable population profiles for all games.

Theorem 13. *A population profile $x \in \Delta_+^N$ is asymptotically stable in the Shapley² replicator dynamic for $v \in \mathbb{V}$ if and only if there exists some player $i \in N$ such that $\mathrm{Sh}_i(v) > \mathrm{Sh}_\ell(v)$ for all $\ell \in N \setminus \{i\}$ and $x = 1_{\{i\}}$.*

A population profile is asymptotically stable in the Shapley^2 replicator dynamic for a game if and only if it is supported by a player i that obtains the unique maximal Shapley payoff in this game—all other types vanish. From this theorem, the following two corollaries are immediate.

Corollary 14. *In the Shapley^2 replicator dynamic, (i) n asymptotically stable population profile generically (see Footnote 3) exists, and*

(ii) asymptotically stable population profiles are robust to small perturbations of the underlying game.

The second part of the corollary says that small "mutations" in the (relative) productivity of the types within a population do not affect the asymptotically stable population profiles in the Shapley^2 replicator dynamic for a game.

7. Concluding remarks

Casajus et al. (2020) have introduced an approach to derive an evolutionary dynamic from an underlying cooperative game with transferable utility. They use a Leontief-type technology, the Lovász extension, to describe the generation of fitness. In this paper, we complete the picture by analyzing dynamics that originate from the other two economically plausible CES technologies: the dual Lovász extension based on the maximum operator and the Shapley extension based on the average operator (see Casajus et al., 2020; Casajus and Kramm, 2021, 2022). The approach allows us to relate the survival of certain types of coalitions to the payoff structure of the cooperative game. For simple monotonic games, we obtain the following in addition to Casajus et al. (2020), who found that if scarce types limit the productivity and competition determines remuneration within partnerships, minimal winning coalitions are the unique asymptotically stable profiles. First, if abundant types limit the productivity and competition determines remuneration within partnerships, as is the case in the dual Lovász-Shapley dynamic, minimal blocking coalitions are the unique asymptotically stable profiles. Second, if all types impact productivity and competition determines remuneration within partnerships, as is the case in the Shapley^2 dynamic, the singleton coalition of the player with the highest Shapley value is the unique asymptotically stable profile.

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References

Algaba, E., Bilbao, J. M., Fernandez, J. R., Jimenez, A., 2004. The Lovász extension of market games. Theory and Decision 56, 229–238.

Casajus, A., 2021. Extension operators for TU games and the Lovász extension. Discrete Applied Mathematics 288, 66–73.

Casajus, A., Kramm, M., 2021. The dual Lovász extension operator and the Shapley extension operator for TU games. Discrete Applied Mathematics 294, 224–232.

Casajus, A., Kramm, M., 2022. The dual Lovász-Shapley value and the Shapley^2 value for non-negatively weighted TU games. HHL Working Paper 196, HHL Leipzig Graduate School of Management, Leipzig, Germany.

Casajus, A., Kramm, M., Wiese, H., 2020. Asymptotic stability in the Lovász-Shapley replicator dynamic for cooperative games. Journal of Economic Theory 186, 104993.

Casajus, A., Wiese, H., 2017. Scarcity, competition, and value. International Journal of Game Theory 46 (2), 295–310.

Filippov, A. F., 1988. Differential Equations With Discontinuous Righthand Sides. Kluwer.

Kannai, Y., 1992. The core and balancedness. In: Aumann, R. J., Hart, S. (Eds.), Handbook of Game Theory With Economic Applications. Vol. 1. North-Holland, Ch. 12, pp. 355–395.

Lovász, L., 1983. Submodular functions and convexity. In: Bachem, A., Gröstschel, M., Korte, B. (Eds.), Mathematical Programming: The State of the Art. Springer, Berlin, pp. 235–257.

Mailath, G. J., 1998. Do people play Nash equilibrium? Lessons from evolutionary game theory. Journal of Economic Literature 36, 1347–1374.

Samuelson, L., 2002. Evolution and game theory. Journal of Economic Perspectives 16, 47–66.

Shapley, L. S., 1953. A value for n-person games. In: Kuhn, H., Tucker, A. (Eds.), Contributions to the Theory of Games. Vol. II. Princeton University Press, Princeton, pp. 307–317.

Sydsaeter, K., Hammond, P., Seierstad, A., Strom, A., 2008. Further Mathematics for Economic Analysis. Pearson Education.

Winter, E., 2002. The Shapley value. In: Aumann, R. J., Hart, S. (Eds.), Handbook of Game Theory With Economic Applications. Vol. 3. North-Holland, Ch. 53, pp. 2025–2054.

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