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The Dual Lovász-Shapley Value and the Shapley^2 Value for Non-Negatively Weighted TU Games

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Abstract

We suggest two economically plausible alternatives to the Lovász-Shapley value for non-negatively weighted TU games (Casajus and Wiese, 2017. Int. J. Game Theory 46 , 295–310), the dual Lovász-Shapley value and the Shapley^2 value. Whereas the former is based on the Lovász extension operator for TU games (Lovász, 1983. Mathematical Programming: The State of the Art, Springer, 235–256; Algaba et al., 2004. Theory Decis. 56, 229–238.), the latter two are based on the dual Lovász extension operator and the Shapley extension operator (Casajus and Kramm, 2021. Discrete Appl. Math. 294, 224–232), respectively.

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1. Introduction

The Shapley value (Shapley, 1953b) probably is the most eminent (single-valued) solution concept for finite cooperative games with transferable utility (TU games). Several weighted generalizations of the Shapley value have been suggested. Shapley (1953a) already suggests the positively weighted Shapley values for TU games that are enriched with positive weights for the players. Kalai and Samet (1987) generalize the positively weighted Shapley value to weight systems, where players are allowed to have zero weights in some sense. Both classes of solutions are efficient, i.e., the players' payoffs sum up to the worth generated by the grand coalition. First, this indicates that the grand coalition is considered to be the productive unit. Second, weights do not affect the generation of worth. That is, weights reflect the players' characteristics such as bargaining skills or responsibilities.

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Casajus and Wiese (2017) suggest the Lovász-Shapley value for non-negatively weighted TU games, that is, TU games that are enriched with non-negative weights for the players. Its intended interpretation is as follows. Players stand for types of agents in a population and the players' weights for their respective subpopulation sizes. This interpretation is supported by showing that the l payoffs for the Lovász-Shapley value can be obtained from infinite games in which the population of agents is modelled explicitly. In particular, the Lovász extension operator (Lovász, 1983; Algaba et al., 2004) is used to construct vector measure games (see for example Neyman, 2002, Section 3) for any non-negatively weighted TU game. In these games, the payoff of the subpopulation of agents of a particular type under the Mertens value (Mertens, 1988) coincides with the payoff of this type in the underlying non-negatively weighted TU game under the Lovász-Shapley value. Moreover, the types' payoffs sum up to the worth generated by their weights in the Lovász extension of the TU game. That is, the weights affect the worth generated by and distributed among the players.

Interpreting the types' payoffs under the Lovász-Shapley value as the fitness of the type subpopulations, Casajus et al. (2020) derive replicator dynamics from TU games and study the asymptotic stability of populations. Their stability results crucially depend on the technology embodied in the Lovász extension operator, the minimum operator, used to define and construct the Lovász-Shapley value. Later on, Casajus (2021) justifies the use of the Lovász extension operator by showing that it is the unique (proper) extension operator that satisfies some economically plausible properties. Moreover, he considers alternative CES (constant elasticity of substitution) technologies to construct extension operators. Only two of these lead to (weak) extensions that also satisfy these economically plausible properties, the maximum operator and the average operator. Casajus and Kramm (2021) introduce and characterize the dual Lovász extension operator and the Shapley extension operator based on the two operators, respectively.

In this paper, we employ the dual Lovász extension operator and the Shapley extension operator instead of the Lovász extension operator to construct vector measure games analogous to those mentioned above. The payoffs of the type subpopulations in these games under der Mertens value or the Aumann-Shapley value (Aumann and Shapley, 1974) give rise to the *dual Lovász-Shapley value* (Equation 19 and Theorem 5) and the *Shapley*^{2} *value* (Equation 24 and Theorem 10), respectively. We axiomatically characterize both solutions in a similar fashion as Casajus and Wiese (2017) characterize the Lovász-Shapley value (Theorems 6 and 11).

This paper is organized as follows. In the second section, we provide basic definitions and notation. In the third section, we survey the Lovász-Shapley value. In the fourth section, we introduce the dual Lovász-Shapley value. In the fifth section, we introduce the Shapley^2 value. Some remarks conclude the paper. The Appendix contains all proofs.

2. Basic definitions and notation

2.1. Finite games

A **TU game** for a non-empty and finite player set N is given by a **coalition function** $v : 2^N \to \mathbb{R}$, $v(\emptyset) = 0$, where 2^N denotes the power set of N. Subsets of N are called

coalitions; $v(S)$ is called the worth of coalition S. The set of all games for N is denoted by $\mathbb{V}$. In the following, we refer to the members of $\mathbb{V}$ as **games**. A coalition $C \subseteq N$ is called a **carrier** of $v \in \mathbb{V}$, if $v(S) = v(S \cap C)$ for all $S \subseteq N$. A coalition $P \subseteq N$ is called a **partnership** in $v \in \mathbb{V}$ if we have $v(S \cup T) = v(T)$ for all $S \subseteq N \setminus P$ and $T \subsetneq N \setminus P$ (Kalai and Samet, 1987). Players $i, j \in N$ are called **symmetric** in $v \in \mathbb{V}$, if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$. Player $i \in N$ is called a **null player** in $v \in \mathbb{V}$, if $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N \setminus \{i\}$.

For $v, w \in \mathbb{V}$, and $\alpha \in \mathbb{R}$, the games $v + w \in \mathbb{V}$ and $\alpha \cdot v \in \mathbb{V}$ are given by $(v + w)(S) = v(S) + w(S)$ and $(\alpha \cdot v)(S) = \alpha \cdot v(S)$ for all $S \subseteq N$. For $v \in \mathbb{V}$, its **dual** $v^* \in \mathbb{V}$ is given by $v^*(S) = v(N) - v(N \setminus S)$. The **null game** $\mathbf{0} \in \mathbb{V}$ is given by $\mathbf{0}(S) = 0$ for all $S \subseteq N$. For $T \subseteq N$, $T \neq \emptyset$, the game $u_T \in \mathbb{V}$ given by $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise is called a **unanimity game**. Any $v \in \mathbb{V}$ can be uniquely represented by unanimity games. In particular, we have

$$v = \sum_{T \subseteq N : T \neq \emptyset} \lambda_T(v) \cdot u_T, \tag{1}$$

where the coefficients $\lambda_T(v)$ are known as the Harsanyi dividends (Harsanyi, 1959) and can be determined recursively by

$$\lambda_T(v) := v(T) - \sum_{S \subsetneq T : S \neq \emptyset} \lambda_S(v). \tag{2}$$

A **rank order** for N is a bijection $\rho : N \to \{1, 2, \ldots, |N|\}$ with the interpretation that i is the $\rho(i)$th player in ρ; the set of rank orders of N is denoted by R. The set of players before i in ρ is denoted by $B_i(\rho) = \{\ell \in N : \rho(\ell) < \rho(i)\}$. The **marginal contribution** of i in ρ and v is denoted by

$$MC_i^v(\rho) := v(B_i(\rho) \cup \{i\}) - v(B_i(\rho)). \tag{3}$$

A game $v \in \mathbb{V}$ is called **monotonic** if $MC_i^v(\rho) \geq 0$ for all $\rho \in R$ and $i \in N$.

A **solution** (value) for $\mathbb{V}$ is a mapping $\varphi : \mathbb{V} \to \mathbb{R}^N$ assigning a payoff $\varphi_i(v)$ to any player $i \in N$ in any game $v \in \mathbb{V}$. The **Shapley value** (Shapley, 1953b), Sh, is given by

$$\mathrm{Sh}_i(v) := \sum_{\rho \in R} \frac{1}{|R|} \cdot MC_i^v(\rho) = \sum_{T \subseteq N : i \in T} \frac{\lambda_T(v)}{|T|} \qquad \text{for all } v \in \mathbb{V} \text{ and } i \in N. \tag{4}$$

Let $\mathbb{E} := \left\{ f : \mathbb{R}_+^N \to \mathbb{R} \right\}$; the members of $\mathbb{R}_+^N$ are called **resource vectors**; the members of $\mathbb{E}$ are called **resource games** for N. The set $\mathbb{E}$ is a linear space on the reals in the obvious sense. For all $f, g \in \mathbb{E}$ and $\alpha \in \mathbb{R}$, the resource games $f + g$ and $\alpha \cdot f$ are given by $(f + g)(s) = f(s) + g(s)$ and $(\alpha \cdot f)(s) = \alpha \cdot f(s)$ for all $s \in \mathbb{R}_+^N$, respectively. An **extension operator** is a mapping $E : \mathbb{V} \to \mathbb{E}$, $v \mapsto Ev$ that is linear and its extensions are positively homogenous; Ev is called the **extension** of v. An extension operator in the sense of Algaba et al. (2004) additionally satisfies an extension property. We call such extension operators **proper**.

The **Lovász extension operator** (Lovász, 1983; Algaba et al., 2004) $\mathrm{L} : \mathbb{V} \to \mathbb{E}$, $v \mapsto \mathrm{L}v$ for all $v \in \mathbb{V}$ can be expressed in terms of Harsanyi dividends Algaba et al. (2004, Theorem 5). For all $v \in \mathbb{V}$, we have

$$\mathrm{L}v\,(s) := \sum_{T \subseteq N : T \neq \emptyset} \min\nolimits_T (s) \cdot \lambda_T (v) \qquad \text{for all } s \in \mathbb{R}^N_+, \tag{5}$$

where

$$\min\nolimits_T (s) := \min\nolimits_{i \in T} s_i \qquad \text{for all } s \in \mathbb{R}^N_+. \tag{6}$$

Casajus and Wiese (2017, Equations 10 and 11) express the Lovász extensions in terms marginal contributions. For $s \in \mathbb{R}^N_+$, let

$$R\,(s) := \{\rho \in R \mid \rho\,(i) < \rho\,(j) \text{ for all } i, j \in N \text{ with } s_i > s_j\}, \tag{7}$$

i.e., $R\,(s)$ contains those rank orders for which players with a greater entry in s come before players with a smaller entry. For all $v \in \mathbb{V}$, we have

$$\mathrm{L}v\,(s) = \sum_{i \in N} s_i \cdot MC^v_i\,(\rho) \qquad \text{for all } s \in \mathbb{R}^N_+ \text{ and } \rho \in R\,(s)\,. \tag{8}$$

2.2. Infinite games[1]

The **space of players** is a measurable space $(I, \mathcal{C})$ that is isomorphic to $([0,1], \mathcal{B})$, where $\mathcal{B}$ stands for the Borel subsets of $[0,1]$. Members of I and $\mathcal{C}$ are called players and coalitions, respectively. An **(infinite) game** is a mapping $v : \mathcal{C} \to \mathbb{R}$ such that $v\,(\emptyset) = 0$. A game is **finitely additive** if $v\,(S \cup T) = v\,(S) + v\,(T)$ for all $S, T \in \mathcal{C}$, $S \cap T = \emptyset$. A game is **monotonic** if $v\,(S) \leq v\,(T)$ for all $S, T \in \mathcal{C}$, $S \subseteq T$. A game is of **bounded variation** if it is the difference of two monotonic games. Let FA and BV denote the real linear spaces of finitely additive games and of games of bounded variation, respectively. We have $FA \subseteq BV$.

For any non-empty and finite set N, let $\mathbf{0}_N \in \mathbb{R}^N$ and $\mathbf{1}_N \in \mathbb{R}^N$ be given by $(\mathbf{0}_N)_i = 0$ and $(\mathbf{1}_N)_i = 1$ for all $i \in N$. Moreover, for any vector $\mu = (\mu_i)_{i \in N}$ of measures μ_i on $(I, \mathcal{C})$, let $\mathcal{R}\,(\mu) \subseteq \mathbb{R}^N$ denote the range of μ. A game v is a **vector measure game** if it can be written as $v = f \circ \mu$, where $\mu = (\mu_i)_{i \in N}$ is a vector of measures on $(I, \mathcal{C})$ for some non-empty and finite set N and $f : \mathcal{R}\,(\mu) \to \mathbb{R}$ is such that $f\,(\mathbf{0}_N) = 0$.

An automorphism of $(I, \mathcal{C})$ is a measurable bijective mapping $\Theta : I \to I$ such that Θ^{-1} is measurable. Let $\mathcal{G}$ denote the group of automorphisms of $(I, \mathcal{C})$. Each $\Theta \in \mathcal{G}$ induces a linear mapping $\Theta_* : BV \to BV$ given by $v \mapsto \Theta_* v$ and $(\Theta_* v)\,(S) = v\,(\Theta S)$ for all $S \in \mathcal{C}$. A set of games $Q \subseteq BV$ is called **symmetric** if $\Theta_* Q = Q$ for all $\Theta \in \mathcal{G}$. For any set of games $Q \subseteq BV$, let Q^+ denote its subset of monotonic games. For $Q \subseteq BV$, a mapping $\varphi : Q \to BV$ is called **positive** if $\varphi\,(Q^+) \subseteq BV^+$; **symmetric** if for every $\Theta \in \mathcal{G}$ and $v \in Q$, $\Theta_* v \in Q$ implies $\varphi\,(\Theta_* v) = \Theta_*\,(\varphi v)$; and **efficient** if for every $v \in Q$, $(\varphi v)\,(I) = v\,(I)$.

[1] Definitions and notation in this section closely follow Neyman (2002, Sections 3, 7, and 8) and Casajus and Wiese (2017, Section 2.2).

A distribution of payoffs is represented by a finitely additive game. A **value** on a symmetric subset Q of BV is a linear mapping $\varphi : Q \to FA$ that is symmetric, positive, and efficient.

The **Aumann-Shapley value** (Aumann and Shapley, 1974), AS, is the unique value on the linear span of vector measure games $v = f \circ \mu$ for which f is continuously differentiable and μ_i is a non-atomic with $\mu_i(\mathcal{I}) > 0$ for all $i \in N$; it is given by the following diagonal formula (Aumann and Shapley, 1974, Theorem B; Neyman, 2002, pp. 2141),

$$\text{AS}(f \circ \mu)(C) = \sum_{i \in N} \mu_i(C) \cdot \int_0^1 \left. \frac{\partial f}{\partial x_i} \right|_{t \cdot \mu(\mathcal{I})} dt \qquad \text{for all } C \in \mathcal{C}. \tag{9}$$

The **Mertens value** (Mertens, 1988), Me, is the unique value on the linear span of vector measure games $f \circ \mu$ for which μ is a vector of mutually singular non-atomic probability measures and f is continuous and piecewise linear; it is given by the diagonal formula below (Neyman, 2002, Section 8; Haimanko, 2001).

Let $\bar{M}_\ell^N$ denote the linear space of continuous and piecewise linear functions $f : [0,1]^N \to \mathbb{R}$ such that $f(\mathbf{0}_N) = 0$. For $f \in \bar{M}_\ell^N$, let $f_y(x)$ denote the directional derivative of f at $x \in (0,1)^N$ in the direction of $y \in \mathbb{R}^N \setminus \{\mathbf{0}_N\}$. Moreover, for fixed $x \in (0,1)^N$, let

$$\partial f(x; y; z) := \lim_{\varepsilon \downarrow 0} \frac{f_{y+\varepsilon \cdot z}(x) - f_y(x)}{\varepsilon} \tag{10}$$

denote the directional derivative of the mapping $y \mapsto f_y(x)$ at $y \in \mathbb{R}^N$ in the direction of $z \in \mathbb{R}^N \setminus \{\mathbf{0}_N\}$ and $\partial f(x; y; \mathbf{0}_N) = 0$. For any vector μ of mutually singular non-atomic probability measures and $f \in \bar{M}_\ell^N$, we have

$$\text{Me}(f \circ \mu)(C) = \int_0^1 \partial f(t \cdot \mathbf{1}_N; Y; \mu(C))\, dt \qquad \text{for all } C \in \mathcal{C}, \tag{11}$$

where $Y = (Y_i)_{i \in N}$ is a vector of independent random variables, each with the standard Cauchy distribution.

3. The Lovász-Shapley value[2]

In this section, we provide a survey on the Lovász-Shapley value Casajus and Wiese (2017). A **non-negatively weighted game** for N is a pair $(v, s) \in \mathbb{V} \times \mathbb{R}_+^N$. In view of the intended interpretation, we refer to players as types. A **non-negatively weighted solution** for N is a mapping $\varphi : \mathbb{V} \times \mathbb{R}_+^N \to \mathbb{R}^N$. The **Lovász-Shapley value**, LS, is given by

$$\text{LS}_i(v, s) := |R(s)|^{-1} \cdot \sum_{\rho \in R(s)} s_i \cdot MC_i^v(\rho) \qquad \text{for all } v \in \mathbb{V},\ s \in \mathbb{R}_+^N, \text{ and } i \in N \tag{12}$$

[2] The survey in this section closely follows Casajus and Wiese (2017).

or

$$\mathrm{LS}_i(v,s) = s_i \cdot \sum_{T\subseteq N: i\in \mathrm{argmin}_T(s)} \frac{\lambda_T(v)}{|\mathrm{argmin}_T(s)|} \qquad \text{for all } v\in\mathbb{V},\ s\in\mathbb{R}^N_+, \text{ and } i\in N, \tag{13}$$

where

$$\mathrm{argmin}_T(s) := \{i\in T \mid s_i = \min_T(s)\} \qquad \text{for all } s\in\mathbb{R}^N_+. \tag{14}$$

At its very origin, the Lovász-Shapley value has been defined by infinite games. For an extension operator $E:\mathbb{V}\to\mathbb{E}$, $s\in\mathbb{R}^N_+$, and $v\in\mathbb{V}$, we consider associated vector measure games $Ev\circ\mu^s$, where $\mu^s = (\mu^s_i)_{i\in N}$ is a vector measure with the following properties:

D1 For all $i\in N$, μ^s_i is a non-atomic measure on $(I,\mathcal{C})$.

There are $A_i\in\mathcal{C}$, $i\in N$ with the following properties:

D2 For all $i,j\in N$, $i\neq j$, we have $A_i\cap A_j=\emptyset$.

D3 For all $i\in N$ and $C\in\mathcal{C}$, we have $\mu^s_i(C) = \mu^s_i(A_i\cap C)$.

D4 For all $i\in N$, we have $\mu^s_i(A_i) = s_i$.

The intended interpretation of the game $Ev\circ\mu^s$ is the following: The player set I represents the population of all agents and $\mathcal{C}$ all their possible coalitions. The vector measure μ^s indicates how many agents of the different types are in a coalition. Condition **D1** implies that there are no "big" agents. Coalition A_i contains only (**D2**) and all (**D3**) agents of type i, where the size of the subpopulation of type i is s_i (**D4**). The extension operator E determines the worth generated by a coalition C based on the type subpopulation sizes. Note that we have $\mu^s(I) = s$ and therefore $Ev\circ\mu^s(I) = Ev(s)$.

Using to Lovász extension operator and applying the Mertens value to the vector measure games associated with a non-negatively weighted TU game, one obtains the types' Lovász-Shapley payoffs in this non-negatively weighted TU game as the Mertens payoffs of their type populations in the associated vector measure game.

Theorem 1 (Casajus and Wiese, 2017). *For all $v\in\mathbb{V}$, $s\in\mathbb{R}^N_+$, and $i\in N$, we have $\mathrm{LS}_i(v,s) = \mathrm{Me}(\mathrm{L}v\circ\mu^s)(A_i)$ where $\mathrm{L}v\circ\mu^s$ is a vector measure game associated with v and s that satisfies properties **D1**, **D2**, **D3**, and **D4**.*

Casajus and Wiese (2017) also provide an axiomatic characterization of the Lovász-Shapley value involving four properties of non-negatively weighted solutions.

Lovász efficiency, LE. For all $v\in\mathbb{V}$ and $s\in\mathbb{R}^N_+$, we have $\sum_{i\in N}\varphi_i(v,s) = \mathrm{L}v(s)$.

Lovász efficiency determines how the type populations cooperate and produce worth. In particular, the minimum operators in (5) and (14) indicate that scarce types limit the generation of worth. Moreover, Lovász efficiency requires the total worth generated to be distributed among the types.

Competition within partnerships, CP. For all $v\in\mathbb{V}$, $s\in\mathbb{R}^N_+$, $S\subseteq N$, and $i,j\in S$ such that S is a partnership for v and $s_i > s_j$, we have $\varphi_i(v,s) = 0$.

This property is intended to be imposed together with Lovász efficiency under which scarce types limit the generation of worth. Within a partnership, players are only jointly productive. That is, either all of them are involved or not involved in any creation of gains from cooperation. Hence, competitive remuneration of types should result in non-scarce types obtaining a zero payoff.

Weak symmetry, $\mathbf{S}^{-}$. For all $v \in \mathbb{V}$, $i, j \in N$, and $s \in \mathbb{R}_{+}^{N}$ such that i and j are symmetric in v and $s_i = s_j$, we have $\varphi_i(v, s) = \varphi_j(v, s)$.

Symmetric types are equally productive in a the TU game. Hence, they should be rewarded equally when their agent subpopulations have the same size.

Strong monotonicity in the game, Mo. (Young, 1985). For all $v, w \in \mathbb{V}$ and $i \in N$ such that $v(S \cup \{i\}) - v(S) \geq w(S \cup \{i\}) - w(S)$ for all $S \subseteq N \backslash \{i\}$, we have $\varphi_i(v, s) \geq \varphi_i(w, s)$ for all $s \in \mathbb{R}_{+}^{N}$.

Strong monotonicity requires the types' payoffs to reflect their productivity in the TU game.

Theorem 2 (Casajus and Wiese, 2017). *The Lovász-Shapley value is the unique non-negatively weighted solution that satisfies Lovász efficiency (**LE**), weak symmetry (**S^{-}**), the competition within partnerships property (**CP**), and strong monotonicity in the game (**Mo**).*

4. The dual Lovász-Shapley value

Casajus and Kramm (2021) advocate and motivate the **dual Lovász extension operator** $\mathrm{L}^{*} : \mathbb{V} \to \mathbb{E}$ that is given by

$$\mathrm{L}^{*}v(s) := \sum_{T \subseteq N : T \neq \emptyset} \max_T(s) \cdot \lambda_T(v) \qquad \text{for all } v \in \mathbb{V} \text{ and } s \in \mathbb{R}_{+}^{N}, \tag{15}$$

where

$$\max_T(s) := \max_{i \in T} s_i \qquad \text{for all } s \in \mathbb{R}_{+}^{N}, \tag{16}$$

or equivalently by

$$\mathrm{L}^{*}v(s) = \sum_{i \in N} s_i \cdot MC_i^v(\rho) \qquad \text{for all } \rho \in R^{*}(s), \tag{17}$$

where

$$R^{*}(s) := \{\rho \in R \mid \rho(i) < \rho(j) \text{ for all } i, j \in N \text{ with } s_i < s_j\}. \tag{18}$$

Its relation to the Lovász extension operator and justification for its name is given by the following proposition.

Proposition 3 (Casajus and Kramm, 2021). *For all $v \in \mathbb{V}$, we have $\mathrm{L}^{*}v = \mathrm{L}v^{*}$.*

Using the dual Lovász extension operator in the vector measure games introduced in Section 3, one obtains the non-negatively weighted solution LS^* given by

$$\mathrm{LS}_i^*(v,s) := \mathrm{Me}(\mathrm{L}^*v \circ \mu^s)(A_i) \qquad \text{for all } v \in \mathbb{V} \text{ and } s \in \mathbb{R}_+^N, \tag{19}$$

where $\mathrm{L}^*v \circ \mu^s$ is a vector measure game associated with v and s that satisfies properties **D1**, **D2**, **D3**, and **D4**. As the diagonal formula (11) for the Mertens value indicates and which is shown later on, it is well-defined, that is, it does not depend on the choices made in the construction of the vector measure games $\mathrm{L}^*v \circ \mu^s$. We call this non-negatively weighted solution the **dual Lovász-Shapley value**. The "dual Lovász" part of this name refers to the fact that we use the dual Lovász extension operator in its definition. The "Shapley" part refers to the fact that we use a generalization of the Shapley value to infinite games, the Mertens value.

Theorem 1, Proposition 4, and (19) imply the following relation between the Lovász-Shapley value and the dual Lovász-Shapley value, which adds to the justification of the "dual" in its name.

Proposition 4. *For (20) and (21), we have* $\mathrm{LS}^*(v,s) = \mathrm{LS}(v^*,s)$ *for all* $v \in \mathbb{V}$ *and* $s \in \mathbb{R}_+^N$.

The next theorem shows that the dual Lovász-Shapley value is well-defined and expresses the dual Lovász-Shapley value directly in terms of the non-negatively weighted games. Its proof is referred to Appendix A.

Theorem 5. *For all* $v \in \mathbb{V}$, $s \in \mathbb{R}_+^N$, *and* $i \in N$, *we have*

$$\mathrm{LS}_i^*(v,s) = |R^*(s)|^{-1} \cdot \sum_{\rho \in R^*(s)} s_i \cdot MC_i^v(\rho) \tag{20}$$

and

$$\mathrm{LS}_i^*(v,s) = s_i \cdot \sum_{T \subseteq N : i \in \mathrm{argmax}_T(s)} \frac{\lambda_T(v)}{|\mathrm{argmax}_T(s)|}, \tag{21}$$

where

$$\mathrm{argmax}_T(s) := \{i \in T \mid s_i = \max_T(s)\} \qquad \textit{for all } s \in \mathbb{R}_+^N. \tag{22}$$

The dual Lovász-Shapley value can be characterized analogously to the Lovász-Shapley value (Theorem 2). Instead of Lovász efficiency and the competition within partnerships property, one employs dual versions of these properties.

Dual Lovász efficiency, LE*. For all $v \in \mathbb{V}$ and $s \in \mathbb{R}_+^N$, we have $\sum_{i \in N} \varphi_i(v,s) = \mathrm{L}^*v(s)$.

Dual Lovász efficiency determines how the type populations cooperate and produce worth. In particular, the maximum operators in (21) and (22) indicate that abundant types limit the generation of worth. Moreover, dual Lovász efficiency requires the total worth generated to be distributed among the types.

Dual competition within partnerships, CP*. For all $v \in \mathbb{V}$, $s \in \mathbb{R}_+^N$, $S \subseteq N$, and $i, j \in S$ such that S is a partnership for v and $s_i > s_j$, we have $\varphi_j(v, s) = 0$.

This property is intended to be imposed together with dual Lovász efficiency under which abundant types limit the generation of worth. Within a partnership, players are only jointly productive. That is, either all of them are involved or not involved in any creation of gains from cooperation. Hence, competitive remuneration of types should result non-abundant types to obtain a zero payoff.

Theorem 6. *The dual Lovász-Shapley value is the unique non-negatively weighted solution that satisfies dual Lovász efficiency (**LE***), weak symmetry (**S$^-$**), the dual competition within partnerships property (**CP***), and strong monotonicity in the game (**Mo**).*

The proof of the theorem is referred to Appendix B.[3] This characterization of dual Lovász-Shapley value is non-redundant for $|N| > 1$. The proof of this claim is referred to Appendix C.

The following remarks provide some basic properties of the dual Lovász-Shapley value that are rather immediate from its formulas (20) or (21).

Remark 7. The dual Lovász-Shapley value is a generalization of the Shapley value. For $s \in \mathbb{R}_+^N$ such that $s_i = 1$ for all $i \in N$, we have $\mathrm{LS}^*(v, s) = \mathrm{Sh}(v)$ for all $v \in \mathbb{V}$.

Remark 8. The dual Lovász-Shapley value is not additive in the weight vector, but positively homogenous. Moreover, it is continuous in the game but not in the weights.

Remark 9. The dual Lovász-Shapley value can be expressed in a particularly simple way for generic weight vectors, that is, any two players have different weights. For $\rho \in R$, set

$$\mathbb{R}_+^{N\,*}(\rho) := \left\{ s \in \mathbb{R}_+^N \mid \text{for all } i, j \in N,\ \rho(i) < \rho(j) \text{ implies } s_i < s_j \right\}.$$

Then, we have

$$\mathrm{LS}_i^*(v, s) = s_i \cdot MC_i^v(\rho)$$

for all $\rho \in R$, $v \in \mathbb{V}$, $i \in N$, and $s \in \mathbb{R}_+^{N\,*}(\rho)$.

[3]One easily checks that strong monotonicity in the game can be replaced by a weaker requirement in Theorem 6, marginality in the game: For all $v, w \in \mathbb{V}$, $i \in N$, and $s \in \mathbb{R}_+^N$ such that $v(S \cup (i)) - v(S) = w(S \cup (i)) - w(S)$ for all $S \subseteq N \setminus \{i\}$, we have $\varphi_i(v, s) = \varphi_i(w, s)$. For $T \subseteq N$, $T \neq \emptyset$, the coalition T is a partnership in u_T. Using this fact, it is straightforward to show that strong monotonicity in the game can be replaced by additivity in the game and the null player property in Theorem 6. Additivity in the game: For all $v, w \in \mathbb{V}$ and $s \in \mathbb{R}_+^N$, we have $\varphi(v + w, s) = \varphi(v, s) + \varphi(w, s)$. Null player property: For all $v \in \mathbb{V}$, $i \in N$, and $s \in \mathbb{R}_+^N$ such that i is a null player in v, we have $\varphi_i(v, s) = 0$.

5. The Shapley² value

Casajus and Kramm (2021) advocate and motivate the **Shapley extension operator** $\mathrm{S} : \mathbb{V} \to \mathbb{E}$ that is given by

$$\mathrm{S}v(s) := \sum_{T \subseteq N : T \neq \emptyset} \left[\frac{\sum_{\ell \in T} s_\ell}{|T|} \right] \cdot \lambda_T(v) = \sum_{i \in N} s_i \cdot \mathrm{Sh}_i(v) \qquad \text{for all } v \in \mathbb{V} \text{ and } s \in \mathbb{R}_+^N. \quad (23)$$

Using the Shapley extension operator in the vector measure games introduced in Section 3, one obtains the non-negatively weighted solution SS given by

$$\mathrm{SS}_i(v, s) := \mathrm{AS}(\mathrm{S}v \circ \mu^s)(A_i) \qquad \text{for all } v \in \mathbb{V} \text{ and } s \in \mathbb{R}_+^N, \quad (24)$$

where $\mathrm{S}v \circ \mu^s$ is a vector measure game associated with v and s that satisfies properties **D1**, **D2**, **D3**, and **D4**. AS the diagonal formula (9) for the Aumann-Shapley value indicates and which is shown later on, it is well-defined, that is, it does not depend on the choices made in the construction of the vector measure games $\mathrm{S}v \circ \mu^s$. We call this non-negatively weighted solution the **Shapley² value**, where the power of two refers to both to the fact that we use the Shapley extension operator in its definition and that we also use a generalization of the Shapley value to infinite games, the Aumann-Shapley value. This is reflected by the symbol we use to represent it, SS. The next theorem shows that the Shapley² value is well-defined and expresses the Shapley² value directly in terms of the non-negatively weighted games. Its proof is referred to Appendix D.

Theorem 10. *For all $v \in \mathbb{V}$, $s \in \mathbb{R}_+^N$, and $i \in N$, we have*

$$\mathrm{SS}_i(v, s) = s_i \cdot \mathrm{Sh}_i(v). \quad (25)$$

The Shapley² value can be characterized analogously to Lovász-Shapley value (Theorem 2) and the dual Lovász-Shapley value (Theorem 6).

Shapley efficiency, SE. For all $v \in \mathbb{V}$ and $s \in \mathbb{R}_+^N$, we have $\sum_{i \in N} \varphi_i(v, s) = \mathrm{S}v(s)$.

Shapley efficiency determines how the type populations cooperate and produce worth. Moreover, Shapley efficiency requires the total worth generated to be distributed among the types.

Proportionality within partnerships, PP. For all $v \in \mathbb{V}$, $s \in \mathbb{R}_+^N$, $S \subseteq N$, and $i, j \in S$ such that S is a partnership for v, we have $\varphi_i(v, s) \cdot s_j = \varphi_j(v, s) \cdot s_i$.

This property is intended to be imposed together with Shapley efficiency under which the generation of worth is proportional to the types weights. Within a partnership, players are only jointly productive. Hence, competitive remuneration of types should result in payoffs proportional to types' weights.

Theorem 11. *The Shapley² value is the unique non-negatively weighted solution that satisfies Shapley efficiency (**SE**), weak symmetry (**S^-**), proportionality within partnerships (**PP**), and strong monotonicity in the game (**Mo**).*

The proof of the theorem is referred to Appendix E.[4] This characterization of Shapley^2 value is non-redundant for $|N| > 1$. The proof of this claim is referred to Appendix F.

The following remarks provide some basic properties of the Shapley^2 value that are rather immediate from its formula (25) and well-known properties of the Shapley value for TU games.

Remark 12. The Shapley^2 value is a generalization of the Shapley value. For $s \in \mathbb{R}_+^N$ such that $s_i = 1$ for all $i \in N$, we have $\mathrm{SS}(v, s) = \mathrm{Sh}(v)$ for all $v \in \mathbb{V}$.

Remark 13. The Shapley^2 value is both additive and positively homogenous in the weight vector. Moreover, it is continuous both in the game and in the weights.

6. Concluding remarks

In this paper, we advocated two economically reasonable alternatives to the Lovász-Shapley value for non-negatively weighted TU games, the dual Lovász-Shapley value and the Shapley^2 value. These could be used to derive replicator dynamics from TU games and to generate stability results in the spirit of Casajus et al. (2020).

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Appendix A. Proof of Theorem 5

We first show that (the right-hand sides of) (20) and (21) coincide. Both are linear in the game. Moreover, the duality mapping $\mathbb{V} \to \mathbb{V}$, $v \mapsto v^*$ also is linear. Hence, it suffices to show the claim for unanimity games. Let $T \subseteq N$, $T \neq \emptyset$, $s \in \mathbb{R}_+^N$, and $i \in N$. For u_T and $\rho \in R^*(s)$, all marginal contributions are zero except that of the last player from T in ρ, which is one. By (18), this last player is a member of $\operatorname{argmax}_T s$. Hence, we obtain

$$\mathrm{LS}_i^*(u_T, s) \overset{(20)}{=} \begin{cases} \frac{\max_T s}{|\operatorname{argmax}_T s|}, & i \in \operatorname{argmax}_T s \\ 0, & i \in N \setminus \operatorname{argmax}_T s \end{cases} \overset{(21)}{=} \mathrm{LS}_i^*(u_T, s), \tag{A.1}$$

and we are done.

Remains to show

$$\mathrm{Me}(\mathrm{L}^* v \circ \mu^s)(A_i) = s_i \cdot \sum_{T \subseteq N : i \in \operatorname{argmax}_T(s)} \frac{\lambda_T(v)}{|\operatorname{argmax}_T(s)|} \tag{A.2}$$

[4] One easily checks that strong monotonicity in the game can be replaced with marginalityin the game or additivity in the game and the null player property in Theorem 11 (see Footnote 3).

for all $v \in \mathbb{V}$, $s \in \mathbb{R}_+^N$, and $i \in N$ and all $\mathrm{L}^* v \circ \mu^s$ and all vector measure games associated with v and s that satisfy properties **D1**, **D2**, **D3**, and **D4**. Whereas not being differentiable, the dual Lovász extensions are piecewise linear. Hence, the diagonal formula for the Mertens value (11) applies to the vector measure games in (19). In order to prove (A.2), one only needs to slightly adjust the proof of Casajus and Wiese (2017, Theorem 8). It is not difficult to check that the proof runs through smoothly when one replaces the operators $\min_T$ and argmin_T with $\max_T$ and argmax_T, $T \subseteq N$, $T \neq \emptyset$ and adjusts the reasoning accordingly.

Appendix B. Proof of Theorem 6

In the following, one uses the formulas for LS^* from Theorem 1. By (15), (16), (21), and (22), LS^* satisfies **LE***. By (20), LS^* satisfies **Mo**. For the rest of the proof, one only needs to slightly adjust the proof of Casajus and Wiese (2017, Theorem 1). It is not difficult to check that the proof runs through smoothly when one replaces the operators argmin_T with argmax_T, $T \subseteq N$, $T \neq \emptyset$ and adjusts the reasoning accordingly.

Appendix C. Non-redundance in Theorem 6

In order to prove this claim, one only needs to slightly adjust the proof of non-redundancy in Casajus and Wiese (2017, Theorem 1). It is not difficult to check that the proof runs through smoothly when one replaces the operators $\min_T$ and argmin_T with $\max_T$ and argmax_T, $T \subseteq N$, $T \neq \emptyset$, L with L^*, $R(s)$ with $R^*(s)$, and adjusts the reasoning accordingly.

Appendix D. Proof of Theorem 10

Since the Shapley extensions are differentiable, the diagonal formula for the Aumann-Shapley value (9) applies to the vector measure games in (24). Moreover, the Shapley extensions $\mathrm{S}v$, $v \in \mathbb{V}$ are linear. Hence, it suffices to show the claim for unanimity games. Let $T \subseteq N$, $T \neq \emptyset$, $s \in \mathbb{R}_+^N$, $i \in T$, and the vector measure μ^s be as in the theorem. We obtain

$$\begin{aligned}
\mathrm{SS}_i(u_T, s) &\overset{(24)}{=} \mathrm{AS}(\mathrm{S}u_T \circ \mu^s)(A_i) \\
&\overset{(9)}{=} \sum_{i \in N} \mu_i^s(A_i) \cdot \int_0^1 \left.\frac{\partial \mathrm{S}u_T}{\partial x_i}\right|_{t \cdot \mu^s(\mathcal{I})} dt \\
&\overset{\mathbf{D2},\mathbf{D3},\mathbf{D4}}{=} s_i \cdot \int_0^1 \left.\frac{\partial \mathrm{S}u_T}{\partial x_i}\right|_{t \cdot s} dt \\
&\overset{(23)}{=} s_i \cdot \int_0^1 \left.\frac{\partial \left(\frac{\sum_{\ell \in T} x_\ell}{|T|}\right)}{\partial x_i}\right|_{t \cdot s} dt \\
&= s_i \cdot \int_0^1 \left.\frac{1}{|T|}\right|_{t \cdot s} dt = \frac{s_i}{|T|} \overset{(1),(4)}{=} s_i \cdot \mathrm{Sh}_i(u_T).
\end{aligned}$$

If $i \in N \setminus T$, we have $\partial \mathrm{S}u_T / \partial x_i = 0$ and therefore $\mathrm{AS}\left(\mathrm{S}u_T \circ \mu^s\right)(A_i) = 0 = s_i \cdot \mathrm{Sh}_i\left(u_T\right),$ which concludes the proof.

Appendix E. Proof of Theorem 11

In the following, we use the formula for SS from Theorem 10. Existence: The non-negatively weighted solution SS inherits **Mo** from Sh (Young, 1985). It is well-known that Sh is symmetric: for all $i, j \in N$ and $v \in \mathbb{V}$ such that i and j are symmetric in v, we have $\mathrm{Sh}_i(v) = \mathrm{Sh}_j(v)$. This implies that SS satisfies $\mathbf{S}^-$. Any two players in a partnership are symmetric. This implies that SS satisfies **PP**. Property **SE** is immediate from its definition.

Uniqueness: We mimic the basic idea of Young's (1985, Theorem 2) proof. Let the non-negatively weighted value φ obey **SE**, $\mathbf{S}^-$, **PP**, and **Mo**. For all $v \in \mathbb{V}$, set

$$\mathcal{T}(v) := \{T \subseteq N \mid T \neq \emptyset \text{ and } \lambda_T(v) \neq 0\}. \tag{E.1}$$

We show $\varphi = \mathrm{SS}$ by induction on $|\mathcal{T}(v)|$.

Induction basis: Let $|\mathcal{T}(v)| = 0$, i.e., $v = \mathbf{0}$. The coalition N is a partnership for $\mathbf{0}$. For $s \in \mathbb{R}^N_+$ such that $s_\ell = 0$ for all $\ell \in N$, we have $\varphi_i(\mathbf{0}, s) = 0 = \mathrm{SS}_i(\mathbf{0}, s)$ for all $i \in N$ by **SE** and $\mathbf{S}^-$. Let now $s \in \mathbb{R}^N_+$ be such that $s_k > 0$ for some $k \in N$. By **PP**, we have

$$s_i \cdot \varphi_j(\mathbf{0}, s) = s_j \cdot \varphi_i(\mathbf{0}, s) \qquad \text{for all } i, j \in N. \tag{E.2}$$

Summing up (E.2) over $j \in N$ gives

$$0 \overset{\mathbf{SE}}{=} s_i \cdot \sum_{j \in N} \varphi_j(\mathbf{0}, s) \overset{\text{(E.2)}}{=} \varphi_i(\mathbf{0}, s) \cdot \sum_{j \in N} s_j.$$

Since $\sum_{j \in N} s_j > 0$, we have $\varphi_i(\mathbf{0}, s) = 0 = \mathrm{SS}_i(\mathbf{0}, s)$ for all $i \in N$.

Induction hypothesis (IH): Suppose $\varphi(v, s) = \mathrm{SS}(v, s)$ for all $v \in \mathbb{V}$ and $s \in \mathbb{R}^N_+$ such that $|\mathcal{T}(v)| \leq t$.

Induction step: Fix now $v \in \mathbb{V}$ and $s \in \mathbb{R}^N_+$ such that $|\mathcal{T}(v)| = t + 1$. Set $T(v) := \bigcap_{T \in \mathcal{T}(v)} T$. Let $i \in N \setminus T(v)$ and $T \in \mathcal{T}(v)$ such that $i \in N \setminus T$. Set $w := v - \lambda_T(v) \cdot u_T$. We have $v(S \cup \{i\}) - v(S) = w(S \cup \{i\}) - w(S)$ for all $S \subseteq N \setminus \{i\}$ and $|\mathcal{T}(w)| = t$. Hence, we obtain

$$\varphi_i(v, s) \overset{\mathbf{Mo}}{=} \varphi_i(w, s) \overset{IH}{=} \mathrm{SS}_i(w, s) \overset{\mathbf{Mo}}{=} \mathrm{SS}_i(v, s) \qquad \text{for all } i \in N \setminus T(v). \tag{E.3}$$

By construction, $T(v)$ is a partnership for v, which implies that any two players in $T(v)$ are symmetric. If $s_\ell = 0$ for all $\ell \in T(v)$, we obtain

$$\begin{aligned}\varphi_i(v, s) &\overset{\mathbf{SE}, \mathbf{S}^-}{=} \frac{\mathrm{S}v(s) - \sum_{\ell \in N \setminus T(v)} \varphi_\ell(v, s)}{|T(v)|} \\ &\overset{\text{(E.3)}}{=} \frac{\mathrm{S}v(s) - \sum_{\ell \in N \setminus T(v)} \mathrm{SS}_\ell(v, s)}{|T(v)|} \\ &\overset{\mathbf{SE}, \mathbf{S}^-}{=} \mathrm{SS}_i(v, s)\end{aligned} \tag{E.4}$$

for all $i \in T(v)$.

Let now $s \in \mathbb{R}_{+}^{N}$ be such that $s_k > 0$ for some $k \in T(v)$. By **PP**, we have

$$s_i \cdot \varphi_j(v,s) = s_j \cdot \varphi_i(v,s) \qquad \text{for all } i,j \in T(v). \tag{E.5}$$

Summing up (E.5) over $j \in T(v)$ gives

$$s_i \cdot \sum_{j \in T(v)} \varphi_j(v,s) = \varphi_i(v,s) \cdot \sum_{j \in T(v)} s_j. \tag{E.6}$$

Since $\sum_{j \in T(v)} s_j > 0$, we obtain

$$\begin{aligned}
\varphi_i(v,s) &\overset{(E.6)}{=} \frac{s_i}{\sum_{j \in T(v)} s_j} \cdot \sum_{j \in T(v)} \varphi_j(v,s) \\
&\overset{\mathbf{SE},(E.3),(E.4)}{=} \frac{s_i}{\sum_{j \in T(v)} s_j} \cdot \left(\mathrm{S}v(s) - \sum_{\ell \in N \setminus T(v)} \mathrm{SS}_\ell(v,s) \right) \\
&\overset{(25)}{=} \mathrm{SS}_i(v,s),
\end{aligned}$$

which concludes the proof.

Appendix F. Non-redundance in Theorem 11

The non-negatively weighted solution $\varphi^{\mathbf{SE}}$ given by $\varphi_i^{\mathbf{SE}}(v,s) := 0$ for all $v \in \mathbb{V}$, $s \in \mathbb{R}_{+}^{N}$, and $i \in N$ satisfies all properties but **SE**. Let $\alpha : N \to \mathbb{R}$ be such that $\sum_{\ell \in N} \alpha(\ell) = 0$ and $\alpha(i,s) > 0$ for some $i \in N$. The solution $\varphi^{\mathbf{S}^-}$ given by

$$\varphi_i^{\mathbf{S}^-}(v,s) := \begin{cases} \mathrm{SS}_i(v,s), & s_\ell > 0 \text{ for some } \ell \in \mathbb{N}, \\ \alpha(i), & s_\ell = 0 \text{ for all } \ell \in \mathbb{N} \end{cases} \qquad \text{for all } v \in \mathbb{V},\ s \in \mathbb{R}_{+}^{N},\ \text{and } i \in N$$

satisfies all properties but $\mathbf{S}^-$. Let $\beta : N \times \mathbb{R}_{+}^{N} \to \mathbb{R}$ be such that $\sum_{\ell \in N} \beta(\ell,s) = 0$ for all $s \in \mathbb{R}_{+}^{N}$, $\beta(i,s) = \beta(j,s)$ for all $i,j \in N$ and $s \in \mathbb{R}_{+}^{N}$ such that $s_i = s_j$, and $\beta(i,s) > 0$ for some $i \in N$ and $s \in \mathbb{R}_{+}^{N}$. The solution $\varphi^{\mathbf{PP}}$ given by $\varphi_i^{\mathbf{PP}}(v,s) := \mathrm{SS}_i(v,s) + \beta(i,s)$ for all $v \in \mathbb{V}$, $s \in \mathbb{R}_{+}^{N}$, and $i \in N$ satisfies all properties but **PP**. The solution $\varphi^{\mathbf{Mo}}$ given by

$$\varphi_i^{\mathbf{Mo}}(v,s) := \begin{cases} \frac{s_i}{\sum_{\ell \in N} s_\ell} \cdot \mathrm{S}v(s), & \sum_{\ell \in N} s_\ell > 0, \\ 0, & \sum_{\ell \in N} s_\ell = 0 \end{cases} \qquad \text{for all } v \in \mathbb{V},\ s \in \mathbb{R}_{+}^{N},\ \text{and } i \in N$$

satisfies all properties but **Mo**.

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